Q1. (3 points) Determine whether the following can be probability distribution of a discrete random variable $X$ (defined in each case only for the given values of $x$). Please, explain your answer briefly!

(a) 

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>0</th>
<th>1</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.20</td>
<td>0.40</td>
<td>0.39</td>
<td>0.01</td>
</tr>
</tbody>
</table>

$P(x_1) + P(x_2) + P(x_3) + P(x_4) = 0.2 + 0.4 + 0.39 + 0.01 = 1$

and each $p(x) \in [0,1]$

**YES**

(b) $P(X = x) = p(x) = \frac{1}{5}$ for $x = 0, 1, 2, 3, 4, 5$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{5}$</td>
</tr>
</tbody>
</table>

$\sum_{x} p(x) = \frac{6}{5} > 1$

**NO**

(c) $P(X = x) = p(x) = \frac{x+1}{14}$ for $x = -2, 3, 4, 5$

$p(-2) = \frac{-2+1}{14} = -\frac{1}{14} < 0$

**NO**
Q2. (2 points) A box contains three $1 bills, two $5 bills, five $10 bills and one $20 bill. Let $X$ be a random variable equal to the value of a single bill drawn at random from the box.

(a) Find a probability distribution of $X$.

\[
\begin{array}{c|c|c|c|c}
X & 1 & 5 & 10 & 20 \\
\hline
p(x) & \frac{3}{11} & \frac{2}{11} & \frac{5}{11} & \frac{1}{11} \\
\end{array}
\]

(b) Draw a histogram of the probability distribution of $X$.
Q3. (4 points) The probabilities that a building inspector will observe 0, 1, 2, 3 of violations of the building code in a home built in a large development are given in the following table:

<table>
<thead>
<tr>
<th>Number of violations</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.40</td>
<td>0.20</td>
<td>0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Suppose $X$ denotes the number of violations.

(a) Find $P(2 \leq X \leq 4)$

$$P(2 \leq X \leq 4) = P(X=2) + P(X=3) = 0.1 + 0.3 = 0.4$$

(b) Find the mean $\mu$ of the probability distribution of $X$.

$$\mu = 0 \cdot (0.40) + 1 \cdot (0.20) + 2 \cdot (0.1) + 3 \cdot (0.3) = 1.3$$

(c) Find the standard deviation $\sigma$ of the probability distribution of $X$.

$$\sigma^2 = 0^2 \cdot (0.4) + 1^2 \cdot (0.2) + 2^2 \cdot (0.1) + 3^2 \cdot (0.3) - (1.3)^2$$

$$\sigma = \sqrt{1.61} = 1.269$$

(d) Find $P(X \leq \mu - \sigma)$

$$\mu - \sigma = 1.3 - 1.269 \approx 0.03$$

$$P(X \leq 0.03) = P(X=0) = 0.4$$
Q4. (2 points) Suppose $X$ is binomial random variable with parameters $n = 25$ and $p = 0.3$, i.e., $X \sim B(25, 0.3)$. Find the following probabilities.

(a) $P(X > 6)$

\[ n = 25 \quad p = 0.3 \]

\[ P(X > 6) = 1 - P(X \leq 6) = 1 - 0.3407 \]

\[ P(X \geq 7) = 0.6593 \]

(b) $P(X \leq \mu + \sigma)$

\[ \mu = np = 25 \cdot (0.3) = 7.5 \]

\[ \sigma^2 = np(1-p) = 25 (0.3)(0.7) = 5.25 \]

\[ \sigma = \sqrt{5.25} = 2.29 \]

\[ \mu + \sigma = 9.79 \]

\[ P(X \leq 9.79) = P(X \leq 9) = 0.8106 \]
Q5. (2 points) Suppose $X$ is geometric random variable with mean $\mu = 3$. Find the following probabilities

(a) $P(X > 2)$

\[
P = \frac{1}{\mu} = \frac{1}{3}
\]

\[
P(X > 2) = 1 - P(X \leq 2) = 1 - \left( P(X=1) + P(X=2) \right)
\]

\[
= 1 - \left( \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} \right)
\]

\[
= 1 - \frac{5}{9} = \frac{4}{9}
\]

or

\[
P(X > 2) = 1 - P(X \leq 2) = 1 - \left( 1 - \left( 1 - \frac{1}{3} \right)^2 \right)
\]

\[
= \frac{4}{9}
\]

(b) $P(X \leq \mu)$

\[
P(X \leq 3) = 1 - \left( 1 - \frac{1}{3} \right)^3
\]

\[
= \frac{19}{27} \approx 0.7
\]
Q6. (4 points) Suppose $X$ is a uniform variable with $a = 16$ and $b = 40$, i.e., $X \sim U[16, 40]$.

(a) Find $P(X \geq 22)$

\[
\begin{align*}
a &= 16 & \quad b &= 40 \\
X &\sim U[16, 40] \\
P(X \geq 22) &= (40 - 22) \cdot \frac{1}{24} \\
&= \frac{3}{4} = 0.75
\end{align*}
\]

(b) Find the probability that $X$ is more than two standard deviations from the mean.

\[
\begin{align*}
\mu &= \frac{a + b}{2} = 28 \\
\sigma^2 &= \frac{(b-a)^2}{12} = 48 \\
\sigma &= \sqrt{48} = 6.93 \\
P(X < \mu - 2\sigma) + P(X > \mu + 2\sigma) &= P(X < 14.1) + P(X > 41.4) \\
&= 0 + 0 = \boxed{0}
\end{align*}
\]
Q7. (3 points) Let the random variable $Z$ have the standard normal distribution. Find each of the following probabilities.

(a) $P(Z \geq 1.72)$

\[
P(Z \geq 1.72) = 1 - P(Z < 1.72) = 1 - 0.9573 = 0.0427
\]

(b) $P(-2.01 \leq Z \leq 0.48)$

\[
P(-2.01 \leq Z \leq 0.48) = P(Z \leq 0.48) - P(Z \leq -2.01)
\]

\[
= 0.6844 - 0.0222 = 0.6622
\]

(c) $P(-1.24 \leq Z \leq -0.57)$

\[
P(-1.24 \leq Z \leq -0.57) = P(Z \leq -0.57) - P(Z \leq -1.24)
\]

\[
= 0.2843 - 0.1075 = 0.1768
\]
(c) Find a value $c$ such that $P(X < c) = 0.7$

\[ P(X < c) = 0.7 \]
\[
(C - 16) \frac{1}{24} = 0.7 \\
C = 3.28
\]

(d) Find $P(X < 20 | X \leq 30)$

\[
P(X < 20 \mid X \leq 30) = \frac{P(X < 20 \text{ and } X \leq 30)}{P(X \leq 30)}
\]
\[
= \frac{P(X < 20)}{P(X \leq 30)} = \frac{(20 - 16) \frac{1}{24}}{(30 - 16) \frac{1}{24}}
\]
\[
= \frac{2}{7}
\]
Q8. (3 points) Let the random variable $Z$ have the standard normal distribution. Solve each expression for $b$.

(a) $P(Z \geq b) = 0.9744$

$$P(Z \geq b) = 1 - P(Z < b) = 0.9744$$

$\Rightarrow P(Z < b) = 0.0256$

$\Rightarrow b = -1.95$

(b) $P(Z \leq b) = 0.0594$

$\text{[Diagram]}$

$\Rightarrow b = -1.56$

(c) $P(-b \leq Z \leq b) = 0.3030$

$$P(-b \leq Z \leq b) = P(Z \leq b) - P(Z \leq -b) =$$

$$= 1 - 2P(Z \leq -b) = 0.3030$$

$\Rightarrow P(Z \leq -b) = \frac{1 - 0.3030}{2} = 0.3485$

$\Rightarrow -b = -0.39$

$\Rightarrow b = 0.39$
Q9. (2 points) Suppose that the period of time which a tourist spends in a museum is a random variable with $\mu = 73.4$ and $\sigma = 6.8$ minute. Assuming that this random variable is normally distributed, i.e., $X \sim N(73.4, (6.8)^2)$, answer the following questions.

(a) What is the probability that a tourist will spend at most 66 minutes in the museum?

$$
P(X \leq 66) = P\left(Z \leq \frac{66 - 73.4}{6.8}\right)$$

$$
= P(Z \leq -1.09) -$$

$$
= 0.1379$$

(b) What is the probability that a tourist will spend between 45 and 60 minutes in the museum?

$$
P(45 \leq X \leq 60)$$

$$
= P\left(\frac{45 - 73.4}{6.8} \leq Z \leq \frac{60 - 73.4}{6.8}\right)$$

$$
= P(-4.18 \leq Z \leq -1.97)$$

$$
= P(Z \leq -1.97) - P(Z \leq -4.18)$$

$$
= 0.0244 - 0$$

$$
= 0.0244$$