Special values of shifted convolution Dirichlet series

Michael H. Mertens
(joint work with Ken Ono and Kathrin Bringmann)

Emory University

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2 Nuts and bolts
   - Harmonic Maaß forms
   - Rankin-Cohen brackets
   - Poincaré series

3 Holomorphic projection

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Definitions

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ ($k_1 \geq k_2$) with

  $$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n)q^n.$$
**Definitions**

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  \[
  f_i(\tau) = \sum_{n=1}^{\infty} a_i(n)q^n.
  \]

- Rankin-Selberg convolution
  \[
  L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n)a_2(n)}{n^s},
  \]
Definitions

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- Rankin-Selberg convolution

$$L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n)a_2(n)}{n^s},$$

- shifted convolution Dirichlet series (Hoffstein-Hulse, 2013)

$$D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)a_2(n)}{n^s}.$$
We define the shifted convolution Dirichlet series as

\[ D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n + h)a_2(n)(n + h)^\mu}{n^s}. \]

To define the symmetrized shifted convolution Dirichlet series \( \hat{D}^{(\nu)} \), we use

\[ \hat{D}^{(\nu)}(f_1, f_2, h; s), \text{ e.g. for } \nu = 0 \text{ and } k_1 = k_2, \]

\[ \hat{D}^{(0)}(f_1, f_2, h; s) = D^{(f_1, f_2, h; s)} - D^{(f_2, f_1, -h; s)}. \]

The generating function of special values is

\[ L^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1)q^h. \]
derived shifted convolution series

\[ D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n + h)a_2(n)(n + h)^{\mu}0}{n^s}. \]

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\[ \hat{D}^{(0)}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(f_2, f_1, -h; s), \]
Definitions (continued)

- **derived** shifted convolution series

\[
D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n + h)a_2(n)(n + h)^{\mu}0}{n^s}.
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- use to define **symmetrized** shifted convolution Dirichlet series

\[
\hat{D}^{(\mu)}(f_1, f_2, h; s), \text{ e.g. for } \nu = 0 \text{ and } k_1 = k_2,
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\hat{D}^{(0)}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(f_2, f_1, -h; s),
\]

- generating function of special values

\[
\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1)q^h.
\]
A numerical conundrum

\[ L^{(0)}(\Delta, \Delta; \tau) = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots \]
A numerical conundrum

\[ L^{(0)}(\Delta, \Delta; \tau) = -33.383 \ldots q + 266.439 \ldots q^2 - 1519.218 \ldots q^3 + 4827.434 \ldots q^4 - \ldots \]

- define real numbers \( \alpha = 106.10455 \ldots \) and \( \beta = 2.8402 \ldots \), and the weight 12 weakly holomorphic modular form

\[
\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(\tau)(j(\tau)^2 - 1464j(\tau) - \alpha^2 + 1464\alpha),
\]
A numerical conundrum

\[ \mathbb{L}^{(0)}(\Delta, \Delta; \tau) \]

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\]

- play around a bit and find

\[
- \frac{\Delta}{\beta} \left( \frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n \neq 0} r(n)n^{-11} q^n \right)
\]

\[ = -33.383 \ldots q + 266.439 \ldots q^2 - 1519.218 \ldots q^3 + 4827.434 \ldots q^4 - \ldots \]
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Harmonic Maaß forms

**Definition**

Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic function and $k \in \frac{1}{2} \mathbb{Z} \setminus \{1\}$ and $N \in \mathbb{N}$ with

1. $f|_{2-k} \gamma = f$ for all $\gamma \in \Gamma_0(N)$,
2. $\Delta_{2-k} f \equiv 0$ with $\mathbb{H} \ni \tau = x + iy$ and

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

3. $f$ grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

Then $f$ is called a **harmonic Maaß form** of weight $2 - k$ for $\Gamma_0(N)$. The $\mathbb{C}$-vector space of these forms is denoted by $H_{2-k}(\Gamma_0(N))$. 
Lemma

For \( f \in H_{2-k}(\Gamma_0(N)) \) we have the splitting

\[
f(\tau) = \sum_{m=m_0}^{\infty} c_f^+(n) q^n + \frac{(4\pi y)^{1-k}}{k-1} c_f^-(0) + \sum_{\substack{n=n_0 \\ n \neq 0}}^{\infty} c_f^-(n) n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n}.
\]
Proposition (Bruinier-Funke)

\[ \xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M^!_{k}(\Gamma_0(N)), \ f \mapsto 2i y^{2-k} \frac{\partial f}{\partial \tau} \]

is well-defined and surjective with kernel \( M_{2-k}(\Gamma_0(N)) \). Moreover, we have

\[ (\xi_{2-k} f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c^-_f(n) q^n. \]
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\[ (\xi_{2-k}f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c_f(n)q^n. \]

\(-(4\pi)^{1-k}\xi_{2-k}f\) is called the shadow of \( f \).
Proposition (Bruinier-Funke)

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\[ (\xi_{2-k} f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c_f(n)q^n. \]

- \(-(4\pi)^{1-k}\xi_{2-k}f\) is called the shadow of \( f \).
- For \( f_1 \in S_{k_1}(\Gamma_0(N)) \) denote by \( M_{f_1} \) a HMF with shadow \( f_1 \)
Definition

Let $f, g : \mathbb{H} \rightarrow \mathbb{C}$ be smooth functions on the upper half-plane and $k, \ell \in \mathbb{R}$ be some real numbers, the weights of $f$ and $g$. Then for a non-negative integer $\nu$ we define the $\nu$th Rankin-Cohen bracket of $f$ and $g$ by

$$[f, g]_\nu := \frac{1}{(2\pi i)^\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k + \nu - 1}{\nu - \mu} \binom{\ell + \nu - 1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu - \mu} g}{\partial \tau}.$$
Define \( \rho : \mathbb{H} \to \mathbb{C} \) to be smooth functions on the upper half-plane and \( k, \ell \in \mathbb{R} \) be some real numbers, the weights of \( \rho \) and \( \gamma \). Then for a non-negative integer \( \nu \) we define the \( \nu \)th Rankin-Cohen bracket of \( \rho, \gamma \) by

\[
[f, g]_{\nu} := \frac{1}{(2\pi i)^{\nu}} \sum_{\mu=0}^{\nu} (-1)^{\mu} \binom{k + \nu - 1}{\nu - \mu} \binom{\ell + \nu - 1}{\mu} \frac{\partial^{\mu} f}{\partial \tau^{\mu}} \frac{\partial^{\nu - \mu} g}{\partial \tau}.
\]

- \( f, g \) modular of weights \( k, \ell \) \( \Rightarrow \) \( [f, g]_{\nu} \) modular of weight \( k + \ell + 2\nu \).
Poincaré series

- A general Poincaré series of weight $k$ for $\Gamma_0(N)$:

$$\mathbb{P}(m, k, N, \varphi_m; \tau) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} (\varphi_m^* | k \gamma)(\tau),$$

where $\varphi_m^*(\tau) := \varphi_m(y) \exp(2\pi imx)$. 
A general Poincaré series of weight $k$ for $\Gamma_0(N)$:

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two special cases ($m > 0$):

$$P(m, k, N; \tau) := P(m, k, N, e^{-my}; \tau) \in S_k(\Gamma_0(N)),$$

$$Q(-m, k, N; \tau) := P(-m, 2 - k, N, M_{1 - \frac{k}{2}}(-4\pi my); \tau) \in H_{2-k}(\Gamma_0(N))$$

where $M$ is defined in terms of the $M$-Whittaker function.
Lemma

If $k \geq 2$ is even and $m, N \geq 1$, then

$$\xi_{2-k}(Q(-m, k, N; \tau)) = (4\pi)^{k-1} m^{k-1}(k-1) \cdot P(m, k, N; \tau) \in S_k(\Gamma_0(N)).$$
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Idea of holomorphic projection

\[ \Phi : \mathbb{H} \to \mathbb{C} \] continuous, transforming like a modular form of weight \( k \geq 2 \) for some \( \Gamma_0(N) \), moderate growth at cusps (Attention for \( k = 2! \)).
**Idea of holomorphic projection**

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2$!).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$. 

\[ \exists ! \tilde{\Phi} \in S_k(\Gamma_0(N)) \text{ s.t. } \langle \cdot, \Phi \rangle = \langle \cdot, \tilde{\Phi} \rangle \]

This $\tilde{\Phi}$ is (essentially) the holomorphic projection of $\Phi$. The same reasoning works for regularized Petersson inner product $\Rightarrow$ regularized holomorphic projection.
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\(\xrightarrow{\text{same reasoning}}\) regularized Petersson inner product $\Rightarrow$ regularized holomorphic projection.
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Fourier coefficients

**Definition**

If \( \Phi(\tau) = \sum_{n \in \mathbb{Z}} a_{\Phi}(n, y)q^n \), \((y = \text{Im}(\tau))\), then

\[
(\pi_{\text{hol}} \Phi(\tau)) \ := \ (\pi^{(k)}_{\text{hol}} \Phi)(\tau) \ := \ \sum_{n=0}^{\infty} c(n)q^n,
\]

where

\[
c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_{\Phi}(n, y)e^{-4\pi ny}y^{k-2}dy, \quad n > 0.
\]
Properties of holomorphic projection

Proposition

- If \( \Phi \) is holomorphic, then \( \pi_{hol} \Phi = \Phi \).

Remark

For \( k = 2 \), \( \pi_{hol} \Phi \) is a quasi-modular form of weight 2.

For the regularized holomorphic projection, weakly holomorphic forms are possible images.
Properties of holomorphic projection

**Proposition**

- If \( \Phi \) is holomorphic, then \( \pi_{hol} \Phi = \Phi \).
- If \( \Phi \) transforms like a modular form of weight \( k \in \frac{1}{2} \mathbb{Z}, \ k > 2 \), on some group \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \), then \( \pi_{hol} \Phi \in M_k(\Gamma) \).
Properties of holomorphic projection

Proposition

- If $\Phi$ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If $\Phi$ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \text{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.
- The operator $\pi_{hol}$ commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).

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**Remark**

- For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight 2.
- For the regularized holomorphic projection, weakly holomorphic forms are possible images
Holomorphic projection of mixed mock modular forms

Let

\[ G_{a,b}(X,Y) := \sum_{j=0}^{a-2} (-1)^j \binom{a + b - 3}{a - 2 - j} \binom{j + b - 2}{j} X^{a-2-j} Y^j \in \mathbb{C}[X,Y] \]
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Proposition (Zagier)

Let \( f_1 \in S_{k_1}(\Gamma_0(N)) \) and \( f_2 \in S_{k_2}(\Gamma_0(N)) \) be cusp forms as before. Then we have for \( 0 \leq \nu \leq \frac{k_1 - k_2}{2} \)

\[
\pi_{hol}^{reg}([M_{f_1}, f_2]_{\nu})(\tau) = [M_{f_1}^+, f_2]_{\nu}(\tau) - (k_1 - 2)! \sum_{h=1}^{\infty} q^h \left[ \sum_{n=1}^{\infty} a_2(n + h) a_1(n) \right] \\
\times \sum_{\mu=0}^{\nu} \binom{\nu - k_1 + 1}{\nu - \mu} \binom{\nu + k_2 - 1}{\mu} \left( (n + h)^{-\nu - k_2 + 1} G_{2\nu - k_1 + k_2 + 2, k_1 - \mu}(n + h, n) - n^{\mu - k_1 + 1} (n + h)^{\nu - \mu} \right).
\]
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The theorems

**Theorem 1 (M.-Ono)**

If \( 0 \leq \nu \leq \frac{k_1 - k_2}{2} \), then

\[
\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [M_{f_1}^+, f_2]_\nu + F,
\]

where \( F \in \widetilde{M}^{!}_{2\nu+2-k_1+k_2}(\Gamma_0(N)) \). Moreover, if \( M_{f_1} \) is good for \( f_2 \), then \( F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N)) \).
Theorems

**Theorem 1 (M.-Ono)**

If \(0 \leq \nu \leq \frac{k_1 - k_2}{2}\), then

\[
\mathbb{I}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [M^+_f, f_2]_{\nu} + F,
\]

where \(F \in \widetilde{M}^{!}_{2\nu+2-k_1+k_2}(\Gamma_0(N))\). Moreover, if \(M_{f_1}\) is good for \(f_2\), then \(F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))\).

- \(M_{f_1}\) is **good** for \(f_2\) if \([M^+_f, f_2]_{\nu}\) grows at most polynomially at all cusps (very rare phenomenon).
The theorems

Theorem 1 (M.-Ono)

If \( 0 \leq \nu \leq \frac{k_1 - k_2}{2} \), then

\[
\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [M_{f_1}^+, f_2]_\nu + F,
\]

where \( F \in \widetilde{\mathcal{M}}_{2\nu + 2 - k_1 + k_2}(\Gamma_0(N)) \). Moreover, if \( M_{f_1} \) is good for \( f_2 \), then \( F \in \widetilde{\mathcal{M}}_{2\nu + 2 - k_1 + k_2}(\Gamma_0(N)) \).

- \( M_{f_1} \) is good for \( f_2 \) if \( [M_{f_1}^+, f_2]_\nu \) grows at most polynomially at all cusps (very rare phenomenon).
- \( \widetilde{\mathcal{M}}_k(\Gamma_0(N)) \) is the weakly holomorphic extension of

\[
\widetilde{\mathcal{M}}_k(\Gamma_0(N)) = \begin{cases} 
M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\
\mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2.
\end{cases}
\]
An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau)$
An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau)$

$$\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402 \ldots$$
An example

Let \( f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau) \)

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\]

\( Q(-1, 12, 1; \tau) = Q^+(-1, 12, 1; \tau) + Q^-(1, 12, 1; \tau) \in H_{-10}(\text{SL}_2(\mathbb{Z})) \), the canonical preimage of \( P(1, 12, 1; \tau) \) under \( \xi_{-10} \) (up to a constant factor), is good for \( \Delta \)
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\]

\( Q(-1, 12, 1; \tau) = Q^+(-1, 12, 1; \tau) + Q^-(−1, 12, 1; \tau) \in H_{-10}(\text{SL}_2(\mathbb{Z})) \), the canonical preimage of \( P(1, 12, 1; \tau) \) under \( \xi_{-10} \) (up to a constant factor), is good for \( \Delta \)

\[
\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau) - E_2(\tau)}{11! \cdot \beta \cdot \beta} = -33.383 \ldots q + 266.439 \ldots q^2 - 1519.218 \ldots q^3 + 4827.434 \ldots q^4 - \ldots
\]
Theorem 2 (Bringmann-M.-Ono)

Let \( f \in S_k(\Gamma_0(N)) \) be an even weight newform. If \( p \) is a prime with \( p^2 \mid N \), then there exist constants \( \delta_1, \delta_2 \in \mathbb{C} \), a weight 2 weakly holomorphic quasimodular form \( Q_f \in \tilde{M}_2^!(\Gamma_0(N)) \), and a weight \( 2 - k \) weakly holomorphic \( p \)-adic modular form \( \mathcal{L}_f \) for which

\[
\mathcal{L}(f, f; \tau) = \delta_1 f(\tau) \mathcal{L}_f(\tau) + \delta_2 f(\tau) \mathcal{E}_f(\tau) + Q_f(\tau).
\]

Moreover, if \( f \) has complex multiplication, then there are choices with \( \delta_2 = 0 \).
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Let \( f \in S_k(\Gamma_0(N)) \) be an even weight newform. If \( p \) is a prime with \( p^2 \mid N \), then there exist constants \( \delta_1, \delta_2 \in \mathbb{C} \), a weight 2 weakly holomorphic quasimodular form \( Q_f \in \tilde{M}^!_2(\Gamma_0(N)) \), and a weight \( 2 - k \) weakly holomorphic \( p \)-adic modular form \( \mathcal{L}_f \) for which

\[
\mathbb{L}(f, f; \tau) = \delta_1 f(\tau) \mathcal{L}_f(\tau) + \delta_2 f(\tau) \mathcal{E}_f(\tau) + Q_f(\tau).
\]

Moreover, if \( f \) has complex multiplication, then there are choices with \( \delta_2 = 0 \).

- \( \mathcal{E}_F(\tau) \) is the (holomorphic) Eichler integral of \( F(\tau) = \sum_{n \in \mathbb{Z}} A(n) q^n \),

\[
\mathcal{E}_F(\tau) := \sum_{n \neq 0} A(n) n^{1-k} q^n.
\]
Theorem 3 (Bringmann-M.-Ono)

Let $f$ be as in Theorem 2. Then the following are all true.

1. We have that $f$ may be expressed as a finite linear combination of the form

$$f(\tau) = \sum_{p \nmid m} \alpha_m P(m, k, N; \tau),$$

with $\alpha_m \in \mathbb{C}$.

2. In terms of the linear combination in (1), if $Q(\tau) := \sum_{p \nmid m} \frac{\alpha_m}{m^{k-1}} Q(-m, k, N; \tau)$, then

$$\xi_{2-k}(Q) = (4\pi)^{k-1}(k-1)f.$$

3. If $Q^+(\tau) = \sum_n a_Q(n)q^n$, then $a_Q(pn) = 0$ for all $n \in \mathbb{N}$.

4. We have that $D^{k-1}(Q^+) \in M_k^!(\Gamma_0(N))$.
Another example

- Let $f(\tau) = \eta(3\tau)^8 \in S_4^{new}(\Gamma_0(9))$. $f$ has CM by $\mathbb{Q}(\sqrt{-3})$. 
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- Numerics

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- Let

\[
\beta := \frac{(4\pi)^3}{2} \cdot \|P(1, 4, 9)\|^2 = 1.0468\ldots, \quad \gamma := -0.0796\ldots, \\
\delta := -0.8756\ldots. \quad \text{N.B.: } \frac{\delta}{\gamma} = 11
\]
We find by Theorem 1 that

\[ \mathbb{L}(f, f; \tau) = \frac{f(\tau)Q^+(-1, 4, 9; \tau)}{\beta} \]

\[ + \gamma \left( 1 - 24 \sum_{n=1}^{\infty} \sigma_1(3n)q^{3n} \right) + \delta \left( 1 + 12 \sum_{n=1}^{\infty} \sum_{d|3n \text{ } \not|d} dq^{3n} \right). \]
We find by Theorem 1 that

\[ \mathcal{L}(f, f; \tau) = \frac{f(\tau)Q^+(-1, 4, 9; \tau)}{\beta} \]

\[ + \sum_{n=0}^{\infty} b_f(n)q^n \]

[\text{=:}\mathcal{Q}_f(\tau)]

In Theorem 2 we find \( \delta_1 = \frac{1}{\beta} \),

\[ \mathcal{L}_f(\tau) := Q^+(-1, 4, 9; \tau) = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \frac{3}{32}q^8 - \cdots = -E_m(\tau), \]

with

\[ m(\tau) := \left( \frac{\eta(\tau)^3}{\eta(9\tau)^3} + 3 \right)^2 \cdot \eta(3\tau)^8 = q^{-1} + 2q^2 - 49q^5 + 48q^8 + \ldots. \]
Another example (continued)

- $f \mathcal{L}_f$ is a cuspidal 3-adic modular form of weight 2, $f \mathcal{L}_f \equiv 1 \pmod{3}$ as $q$-series.
Another example (continued)

- \( fL_f \) is a cuspidal 3-adic modular form of weight 2, \( fL_f \equiv 1 \pmod{3} \) as \( q \)-series
- This yields

\[
\hat{D}(f, f, h; 3) - b_f(h) \in \frac{3}{\beta} \cdot \mathbb{Z}(3),
\]

\[
\hat{D}(f, f, 9h + 6; 3) - b_f(9h + 6) \in \frac{9}{\beta} \cdot \mathbb{Z}(3),
\]

\[
\hat{D}(f, f, 36h + 30; 3) - b_f(36h + 30) \in \frac{27}{\beta} \cdot \mathbb{Z}(3),
\]

\[\ldots\]
Another example (continued)

- $f \mathcal{L}_f$ is a cuspidal $3$-adic modular form of weight $2$, $f \mathcal{L}_f \equiv 1 \pmod{3}$ as $q$-series
- This yields

\[
\hat{D}(f, f, h; 3) - b_f(h) \in \frac{3}{\beta} \cdot \mathbb{Z}(3),
\]
\[
\hat{D}(f, f, 9h + 6; 3) - b_f(9h + 6) \in \frac{9}{\beta} \cdot \mathbb{Z}(3),
\]
\[
\hat{D}(f, f, 36h + 30; 3) - b_f(36h + 30) \in \frac{27}{\beta} \cdot \mathbb{Z}(3),
\]
\[
\ldots
\]
- the (rational) numbers $\beta(\hat{D}(f, f, h; 3) - b_f(h))$ are ‘almost always’ multiples of any fixed power of $3$. 
Thank you for your attention.