Holomorphic Projection and Mock Modular Forms

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San Antonio, January 11, 2015
1 Introduction
- Mock modular forms
- Holomorphic projection

2 Applications
- Construction of mock modular forms
- Class number type relations for Fourier coefficients
- Shifted convolution $L$-functions and their special values
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   • Construction of mock modular forms
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The modern definition

Definition 1

A mock modular form $f$ of weight $k \in \frac{1}{2} \mathbb{Z} \setminus \{1\}$ for $\Gamma_0(N)$ is the holomorphic part $\mathcal{M}^{+}$ of a harmonic Maass form $\mathcal{M}$, i.e. there is a weakly holomorphic modular form $g \in M_{2-k}^{!}(\Gamma_0(N))$, the shadow of $f$, s.t. $\mathcal{M} = f + g^{*}$ with

$$g^{*}(\tau) := \int_{-\tau}^{\infty} \frac{g(-\bar{z})}{(z + \tau)^{k}} \, dz$$

transforms like a modular form of weight $k$ under $\Gamma_0(N)$. 

Appear in combinatorial $q$-series (e.g. partition ranks) quantum black holes and wall crossing umbral moonshine...
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Idea of holomorphic projection

- \( \Phi : \mathbb{H} \rightarrow \mathbb{C} \) continuous, transforming like a modular form of weight \( k \geq 2 \) for some \( \Gamma_0(N) \), moderate growth at cusps (Attention for \( k = 2! \)).
Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$. 

This $\tilde{\Phi}$ is (essentially) the holomorphic projection of $\Phi$.
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- This $\tilde{\Phi}$ is (essentially) the **holomorphic projection** of $\Phi$.
- same reasoning works for regularized Petersson inner product $\rightsquigarrow$ regularized holomorphic projection.
Fourier coefficients

Definition 2

If \( \Phi(\tau) = \sum_{n \in \mathbb{Z}} a_\Phi(n, y)q^n \), \( (y = \text{Im}(\tau)) \), then

\[
(\pi_{\text{hol}} f)(\tau) := (\pi_{\text{hol}}^k f)(\tau) := \sum_{n=0}^{\infty} c(n)q^n, \text{ where}\]

\[
c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^\infty a_\Phi(n, y)e^{-4\pi ny}y^{k-2}dy, \quad n > 0.
\]
Properties of holomorphic projection

Proposition

- If $\Phi$ is holomorphic, then $\pi_{hol} \Phi = \Phi$.
Proposition

- If $\Phi$ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If $\Phi$ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}, k > 2$, on some group $\Gamma \leq \text{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.

Remark

For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight $2$. For the regularized holomorphic projection, weakly holomorphic forms are possible images.
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- The operator $\pi_{hol}$ commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).
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**Proposition**
- If $\Phi$ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If $\Phi$ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \text{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.
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**Remark**
- For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight 2.
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**Proposition**
- If \( \Phi \) is holomorphic, then \( \pi_{hol} \Phi = \Phi \).
- If \( \Phi \) transforms like a modular form of weight \( k \in \frac{1}{2} \mathbb{Z}, \ k > 2 \), on some group \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \), then \( \pi_{hol} \Phi \in M_k(\Gamma) \).
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**Remark**
- For \( k = 2 \), \( \pi_{hol} \Phi \) is a quasi-modular form of weight 2.
- For the regularized holomorphic projection, weakly holomorphic forms are possible images.
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A modification of holomorphic projection

**Lemma 1 (S. Zwegers)**

For any translation-invariant function $\Phi : \mathbb{H} \to \mathbb{C}$ and $1 < k \in \frac{1}{2}\mathbb{Z}$ we have

$$\pi_{hol}^{(k)}(\Phi)(\tau) = \frac{(k-1)(2i)^k}{4\pi} \int_{\mathbb{H}} \frac{\Phi(z)y^k}{(\tau-z)^k} \frac{dxdy}{y^2},$$  \hspace{1cm} (1)

provided that the right-hand side converges absolutely.
A modification of holomorphic projection

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$$

(1)

provided that the right-hand side converges absolutely.

Lemma 2 (S. Zwegers)

Provided the rhs of (1) converges absolutely for $k \in \frac{1}{2}\mathbb{Z}$, then we have

$$
(\pi_{hol}^{(k)}(\Phi)|_{k\gamma} = \pi_{hol}^{(k)}(\Phi|_{k\gamma})
$$

for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

In particular this holds if $|\Phi(\tau)|y^r$ is bounded on $\mathbb{H}$ for some $r$ and $k > r + 1 > 1$. 
Lemma

Let

\[ \xi_k := 2i y^k \frac{\partial}{\partial \tau}. \]

Then it holds

- \( \xi_{2-k} g^* \doteq g \)
The $\xi$-operator

**Lemma**

Let

$$\xi_k := 2iy^k \frac{\partial}{\partial \tau}.$$  

Then it holds

1. $\xi_{2-k}g^* \equiv g$
2. $(\xi_{2-k}g)|_{k\gamma} = \xi_{2-k}(g|_{2-k\gamma})$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$. 

**Proposition 1 (S. Zwegers)**

Let $\Phi$ be as in Lemma 2. If $\pi(k) \text{hol} \Phi = 0$ and $\xi_k \Phi$ is modular of weight $2-k$ for some $\Gamma_0(N)$, then $\Phi$ is modular of weight $k$. 

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The $\xi$-operator

**Lemma**

Let

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Then it holds

- $\xi_{2-k}g^* = g$
- $(\xi_{2-k}g)|_{k\gamma} = \xi_{2-k}(g|_{2-k\gamma})$ for all $\gamma \in SL_2(\mathbb{Z})$.

**Proposition 1 (S. Zwegers)**

Let $\Phi$ be as in Lemma 2. If $\pi_{hol}^{(k)} \Phi = 0$ and $\xi_k \Phi$ is modular of weight $2 - k$ for some $\Gamma_0(N)$, then $\Phi$ is modular of weight $k$. 
Surjectivity of the shadow map

Proposition (J. H. Bruinier and J. Funke)

Every weakly holomorphic modular form \( g \in \mathcal{M}_k^!(\Gamma_0(N)) \) \((k \neq 1)\) is the shadow of a mock modular form of weight \(2 - k\).
Surjectivity of the shadow map

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Proof.
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Proof.

- multiply the Eichler integral $g^*$ of $g$ by a sufficiently large power of $\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}$, say $h$ with weight $\ell$, to ensure weight and growth conditions.
Proposition (J. H. Bruinier and J. Funke)

Every weakly holomorphic modular form $g \in M^1_k(\Gamma_0(N))$ ($k \neq 1$) is the shadow of a mock modular form of weight $2 - k$.

Proof.

- Multiply the Eichler integral $g^*$ of $g$ by a sufficiently large power of $\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}$, say $h$ with weight $\ell$, to ensure weight and growth conditions.

- By Proposition 1, $M := \pi_{hol}^{(2-k+\ell)}(g^*h) - g^*h$ is modular of weight $2 - k + \ell$ for $\Gamma_0(N)$. 

Surjectivity of the shadow map

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- multiply the Eichler integral \( g^* \) of \( g \) by a sufficiently large power of \( \Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} \), say \( h \) with weight \( \ell \), to ensure weight and growth conditions.
- by Proposition 1, \( \tilde{M} := \pi_{hol}^{(2-k+\ell)} (g^* h) - g^* h \) is modular of weight \(2 - k + \ell\) for \( \Gamma_0(N) \).
- \( \tilde{M} = \frac{1}{h} M + g^* \) is the desired mock modular form.
Class number relations

\[ \sigma_k(n) := \sum_{d | n} d^k, \quad \lambda_k(n) := \frac{1}{2} \sum_{d | n} \min\left(d, \frac{n}{d}\right)^k. \]

\[ \sum_{s \in \mathbb{Z}} H(4n - s^2) + 2\lambda_1(n) = 2\sigma_1(n), \]
Class number relations

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\[ \sum_{s \in \mathbb{Z}} (s^4 - 3ns^2 + n^2) H(4n - s^2) + 2\lambda_5(n) = 0, \]
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Class number relations

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\]

\[ n \text{ odd} \]
\[
\sum_{s \in \mathbb{Z}} H(n - s^2) + \lambda_1(n) = \frac{1}{3} \sigma_1(n)
\]
\[
\sum_{s \in \mathbb{Z}} (4s^2 - n) H(n - s^2) + \lambda_3(n) = 0,
\]
\[
\sum_{s \in \mathbb{Z}} (16s^4 - 12ns^2 + n^2) H(n - s^2) + \lambda_5(n)
\]
\[
= -\frac{1}{12} \sum_{n=x^2+y^2+z^2+t^2} (x^4 - 6x^2y^2 + y^4),
\]
...

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Theorem (D. Zagier)

The function

\[ \mathcal{H}(\tau) := \sum_{n=0}^{\infty} H(n) q^n \]

is a mock modular form of weight \( \frac{3}{2} \) for \( \Gamma_0(4) \). Its shadow is (up to a constant factor) the classical theta function

\[ \vartheta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}. \]
Connection to mock modular forms

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\[ \vartheta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}. \]

All the above relations can be formulated as

\[ c_\nu [\mathcal{H}(\tau), \vartheta]_\nu |U(4) + 2 \sum_{n=1}^{\infty} \lambda_{2\nu+1}(n)q^n \in \begin{cases} \tilde{M}_2(\text{SL}_2(\mathbb{Z})) & \text{if } \nu = 0, \\ S_{2+2\nu}(\text{SL}_2(\mathbb{Z})) & \text{if } \nu > 0. \end{cases} \]
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All the above relations can be formulated as

$$\tilde{c}_\nu[\mathcal{H}(\tau), \vartheta]_\nu | S_{2,1} + \sum_{n=0}^{\infty} \lambda_{2\nu+1} (2n + 1) q^{2n+1} \in \begin{cases} M_2(\Gamma_0(4)) & \text{if } \nu = 0, \\ S_{2+2\nu}(\Gamma_0(4)) & \text{if } \nu > 0. \end{cases}$$
Mock theta functions

Definition 3

A mock modular form \( f \) is called a mock theta function if its shadow is a linear combination of unary theta functions either of the form

\[
\psi_{s,\chi}(\tau) := \sum_{n \in \mathbb{Z}} \chi(n) q^{sn^2}
\]

\((s \in \mathbb{N}, \chi \text{ an even character})\) of weight \( \frac{1}{2} \) (i.e., \( f \) has weight \( \frac{3}{2} \)) or of the form

\[
\theta_{s,\chi}(\tau) := \sum_{n \in \mathbb{Z}} \chi(n) n q^{sn^2}
\]

\((s \in \mathbb{N}, \chi \text{ an odd character})\) of weight \( \frac{3}{2} \) (i.e. \( f \) has weight \( \frac{1}{2} \)).
Let \( f \) be a mock theta function of weight \( \kappa \in \{ \frac{1}{2}, \frac{3}{2} \} \) and \( g \in M_{2-\kappa}(\Gamma) \) be a l.c. of theta functions with \( \Gamma = \Gamma_1(4N) \) for some \( N \in \mathbb{N} \) and fix \( \nu \in \mathbb{N} \). Then there is a finite linear combination \( L_{f,g}^{\nu} \) of functions of the form

\[
\Lambda_{s,t}^{\chi,\psi}(\tau; \nu) = \sum_{r=1}^{\infty} \left( 2 \sum_{\substack{sm^2 - tn^2 = r \\ m,n \geq 1}} \chi(m)\psi(n)(\sqrt{sm} - \sqrt{tn})^{2\nu+1} \right) q^r
\]

\[
+ \psi(0) \sum_{r=1}^{\infty} \chi(r)(\sqrt{sr})^{2\nu+1} q^{sr^2}
\]

with \( s, t \in \mathbb{N} \) and \( \chi, \psi \) are characters as in Definition 3 of conductors \( F(\chi) \) and \( F(\psi) \) respectively with \( sF(\chi)^2, tF(\psi)^2 | N \), such that \( [f, g]_\nu + L_{f,g}^{\nu} \) is a (quasi)-modular form of weight \( 2\nu + 2 \) (possibly weakly holomorphic).
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shifted convolution Dirichlet series (Hoffstein-Hulse, 2013)
\[
D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n + h)a_2(n)}{n^s}.
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**shifted convolution Dirichlet series** (Hoffstein-Hulse, 2013)

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**symmetrized shifted convolution Dirichlet series**

$$\hat{D}^{(0)}(f_1, f_2, h; s) := D(f_1, f_2, h; s) - D(f_2, f_1, -h; s),$$
Notation

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ with
  $$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n)q^n.$$  

- **shifted convolution Dirichlet series** (Hoffstein-Hulse, 2013)
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- **symmetrized shifted convolution Dirichlet series**
  $$\hat{D}(0)(f_1, f_2, h; s) := D(f_1, f_2, h; s) - D(f_2, f_1, -h; s),$$  

- **generating function of special values**
  $$\mathbb{L}(0)(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}(0)(f_1, f_2, h; k_1 - 1)q^h.$$
Notation

- Let \( f_1 \in S_{k_1}(\Gamma_0(N)) \) and \( f_2 \in S_{k_2}(\Gamma_0(N)) \) with
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- shifted convolution Dirichlet series (Hoffstein-Hulse, 2013)
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- symmetrized shifted convolution Dirichlet series
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\hat{D}^{(0)}(f_1, f_2, h; s) := D(f_1, f_2, h; s) - D(f_2, f_1, -h; s),
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\mathbb{L}^{(0)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \hat{D}^{(0)}(f_1, f_2, h; k_1 - 1)q^h.
\]

- There is also a \( \hat{D}^{(\nu)} \) and \( \mathbb{L}^{(\nu)} \) for \( \nu \in \mathbb{N}_0 \) (more complicated).
A numerical conundrum

\[ L^{(0)}(\Delta, \Delta; \tau) = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots \]
A numerical conundrum

\[ L^{(0)}(\Delta, \Delta; \tau) = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots \]

- define real numbers \( \alpha = 106.10455\ldots \) and \( \beta = 2.8402\ldots \), and the weight 12 weakly holomorphic modular form

\[
\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(\tau)(j(\tau)^2 - 1464j(\tau) - \alpha^2 + 1464\alpha),
\]
A numerical conundrum

\[ I^{(0)}(\Delta, \Delta; \tau) = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots \]

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\]

- play around a bit and find

\[
- \frac{\Delta}{\beta} \left( \frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n \neq 0} r(n)n^{-11}q^n \right)
\]

\[ = -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots \]
The theorem

**Theorem 2 (M.-Ono)**

If $0 \leq \nu \leq \frac{k_1 - k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [\mathcal{M}_{f_1}^+, f_2]_\nu + F,$$

where $F \in \tilde{M}^{!}_{2\nu + 2-k_1+k_2}(\Gamma_0(N))$. Moreover, if $\mathcal{M}_{f_1}$ is good for $f_2$, then $F \in \tilde{M}_{2\nu + 2-k_1+k_2}(\Gamma_0(N))$. 

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**Theorem 2 (M.-Ono)**

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- $\mathcal{M}_{f_1}$ is a harmonic Maaß form with shadow $f_1$. $\mathcal{M}_{f_1}$ is good for $f_2$ if $[\mathcal{M}_{f_1}^+, f_2]_\nu$ grows at most polynomially at all cusps (very rare phenomenon).
The theorem

**Theorem 2 (M.-Ono)**

If \( 0 \leq \nu \leq \frac{k_1 - k_2}{2} \), then

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\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1 - 2)!} \cdot [\mathcal{M}^+_f, f_2]_\nu + F,
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where \( F \in \widetilde{M}_2^!\nu + 2 - k_1 + k_2(\Gamma_0(N)) \). Moreover, if \( \mathcal{M}_{f_1} \) is good for \( f_2 \), then \( F \in \widetilde{M}_2^!\nu + 2 - k_1 + k_2(\Gamma_0(N)) \).

- \( \mathcal{M}_{f_1} \) is a harmonic Maa\ß form with shadow \( f_1 \). \( \mathcal{M}_{f_1} \) is good for \( f_2 \) if \( [\mathcal{M}^+_f, f_2]_\nu \) grows at most polynomially at all cusps (very rare phenomenon).
- \( \widetilde{M}_k^!(\Gamma_0(N)) \) is the weakly holomorphic extension of

\[
\widetilde{M}_k(\Gamma_0(N)) = \begin{cases} 
M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\
\mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2.
\end{cases}
\]
An example

Let \( f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau) \)
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\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402 \ldots
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$$Q(-1, 12, 1; \tau) = Q^+(-1, 12, 1; \tau) + Q^-(1, 12, 1; \tau) \in H_{-10}(\text{SL}_2(\mathbb{Z})), \text{ the canonical preimage of } P(1, 12, 1; \tau) \text{ under } \xi_{-10} \text{ (up to a constant factor), is good for } \Delta$$
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\( Q(-1, 12, 1; \tau) = Q^+(-1, 12, 1; \tau) + Q^-(-1, 12, 1; \tau) \in H_{-10}(SL_2(\mathbb{Z})) \), the canonical preimage of \( P(1, 12, 1; \tau) \) under \( \xi_{-10} \) (up to a constant factor), is good for \( \Delta \)

\[
\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta}
\]

\[
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\]
Thank you for your attention.