Note on helicity balance of the Galerkin method for the 3D Navier-Stokes equations

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Abstract

We study the helicity balance of the Galerkin method for the 3D Navier-Stokes equations, and show that although it does not appear to correctly balance helicity in the usual sense, it instead admits a slightly altered helicity balance that matches that of the underlying physics, up to boundary conditions.

1 Introduction

We consider helicity treatment in the commonly used Galerkin discretization for the 3D Navier-Stokes equations (NSE). It is well known that the NSE conserves helicity [13] in the absence of viscous or body forces, and admits a precise helicity balance when these forces are present. This balance is believed critical in flow structure development [12], and is the basis for the cascade of helicity (joint with energy) through the inertial range [4]. Therefore, it is very desirable that the helicity balance be preserved in a numerical scheme. However, in contrast to the energy balance, the helicity balance does not hold in the usual way for the standard Galerkin approximations (see (2.7)), and thus creative and naturally more expensive discretizations have been developed that do so (first in [10] for axisymmetric flow and later in [14] for the full 3D NSE). We show herein that the standard Galerkin method instead balances a slightly altered discrete helicity-type quantity, which is computed as $H = \int_\Omega \mathbf{u}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}$, but $\mathbf{w}$ is the solution of a discrete vorticity equation instead of simply being the curl of the velocity. For any helicity balance to be discretely preserved is rare and naturally of fundamental physical importance, and hence this work provides additional insight into a physical treatment of helicity by this commonly used method.

To better understand the subtleties of the discrete helicity balance, we first recall the derivation of the helicity balance in the continuous case. Begin with the incompressible NSE on a domain $\Omega\times(0,T]$,

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f}, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{1.3}$$

where $\mathbf{u}$ denotes velocity, $\mathbf{u}_0$ the initial velocity, p pressure, and $\mathbf{f}$ body force. We assume further that the body force is potential, $\mathbf{f} = \nabla\varphi$. Applying the rot operator to the momentum equation (1.1) gives the following equation for the vorticity vector $\mathbf{w} := \operatorname{rot}\mathbf{u}$

$$\mathbf{w}_t - \nu\Delta\mathbf{w} + (\mathbf{u}\cdot\nabla)\,\mathbf{w} - (\mathbf{w}\cdot\nabla)\,\mathbf{u} = 0, \tag{1.4}$$

with initial condition $\mathbf{w}(0) = \operatorname{rot}\mathbf{u}_0$.

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Let $V(t)$ be a volume moving with a fluid such that for a given moment $t \geq 0$ the boundary $\partial V(t)$ is a vortex surface, i.e. $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial V(t)$ where $\mathbf{n}$ is a unit normal vector to $\partial V(t)$. The scalar function $h := \mathbf{u} \cdot \mathbf{w}$ denotes *helical density*. It holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} h \, \mathrm{d}\mathbf{x} = \int_{V(t)} h_t + \mathbf{u} \cdot \nabla h \, \mathrm{d}\mathbf{x} = \int_{V(t)} [\mathbf{u}_t \cdot \mathbf{w} + (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{w}] + [\mathbf{w}_t \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla \mathbf{w}) \cdot \mathbf{u}] \, \mathrm{d}\mathbf{x}. \quad (1.5)$$

Multiplying (1.1) with $\mathbf{w}$ and (1.4) with $\mathbf{u}$ and integrating over $V(t)$ one gets due to $\mathbf{w} \cdot \mathbf{n} = 0$ and $\operatorname{div} \mathbf{w} = 0$:

$$\int_{V(t)} \mathbf{u}_t \cdot \mathbf{w} + (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = \nu \int_{V(t)} (\Delta \mathbf{u}) \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} \quad (1.6)$$

$$\int_{V(t)} \mathbf{w}_t \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla \mathbf{w}) \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = \nu \int_{V(t)} (\Delta \mathbf{w}) \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} \quad (1.7)$$

Assume for a moment that the fluid is inviscid, then the righthand sides in (1.6)–(1.7) vanish and so does the righthand side in (1.5), implying $\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} h \, \mathrm{d}\mathbf{x} = 0$. Since under the evolution governed by the Euler equations of an ideal fluid vortex surfaces are "frozen in the fluid" (behave like material surfaces), the condition $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial V(t)$ is satisfied for all $t \geq 0$ if it holds for $t = 0$. This implies that the helicity associated with $V(t)$ is conserved:

$$\int_{V(t)} \mathbf{u}(t) \cdot \mathbf{w}(t) \, \mathrm{d}\mathbf{x} = \int_{V(0)} \mathbf{u}_0 \cdot \operatorname{rot} \mathbf{u}_0 \, \mathrm{d}\mathbf{x}, \qquad \forall\, t \geq 0.$$

The situation becomes more delicate in the viscous case: the viscous terms enter the helicity balance and condition $\mathbf{w} \cdot \mathbf{n} = 0$ is no longer preserved on $\partial V(t)$. The latter means that helicity can be created or destroyed on the boundary of $V(t)$ [11], and thus the helical phenomenon is much more complicated in a viscous fluid.

We shall study the global helicity balance of a viscous flow, setting $V(t) := \Omega$ for all t. For the problem (1.1)–(1.2) to be well-posed one should prescribe a boundary condition on $\partial\Omega$. To analyze the global helicity budget, a natural choice is to assume Ω is a periodic box or a polyhedron with $\mathbf{u} = 0$ on $\partial\Omega$. While the periodic case is of a limited practical interest, homogeneous Dirichlet boundary conditions for velocity describes internal flows well and is a common assumption in numerical analysis of discretization techniques in fluid dynamics. Thus we assume

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega. \quad (1.8)$$

Note that (1.8) implies $\mathbf{w} \cdot \mathbf{n} = 0$. Thus we set $V(t) := \Omega$ in (1.5)–(1.7) and use $\operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{u} = 0$, vector identity $\Delta = \nabla \operatorname{div} - \operatorname{rot} \operatorname{rot}$, and integration by parts to reduce the viscous terms:

$$\int_\Omega (\Delta \mathbf{u}) \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = -\int_\Omega (\operatorname{rot} \operatorname{rot} \mathbf{u}) \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = -\int_\Omega (\operatorname{rot} \mathbf{w}) \cdot (\operatorname{rot} \mathbf{u}) \, \mathrm{d}\mathbf{x} = -\int_\Omega (\operatorname{rot} \operatorname{rot} \mathbf{w}) \cdot \mathbf{u} \, \mathrm{d}\mathbf{x}$$
$$= \int_\Omega (\Delta \mathbf{w}) \cdot \mathbf{u} \, \mathrm{d}\mathbf{x} = -\int_\Omega (\nabla \mathbf{w}) : (\nabla \mathbf{u}) \, \mathrm{d}\mathbf{x}.$$

Integrating over time we get the following NSE helicity balance equation

$$H(T) + 2\nu \int_0^T (\nabla \mathbf{u}(t), \nabla \mathbf{w}(t)) \, dt = H(0), \quad \text{with } H(t) := (\mathbf{u}(t), \mathbf{w}(t)). \quad (1.9)$$

Here and further $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ scalar product, e.g. $(\mathbf{u}, \mathbf{w}) := \int_\Omega \mathbf{u} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}$. We desire numerical schemes to admit some discrete analog of (1.9), as it provides evidence of physical relevance of solutions.

2 Helicity balance for Galerkin approximations

Consider now the Galerkin formulation of the NSE. We assume a finite element method, although other choices of discrete spaces, see e.g. [8], are also suitable for further discussion. Denote inf-sup stable discrete velocity and pressure spaces by $X_h \subset H_0^1(\Omega)^3$ and $Q_h \subset L_0^2(\Omega) := \{p \in$

$L^2(\Omega) \mid \int_\Omega p \,\mathrm{d}\mathbf{x} = 0\}$ respectively (see [3, 6] for examples of such spaces), denote V_h to be the space of discretely divergence free functions in X_h, $V_h = \{\mathbf{v}_h \in X_h : (\operatorname{div} \mathbf{v}_h, q_h) = 0 \ \forall \ q_h \in Q_h\}$, and define the operators P_{V_h} and P_{Q_h} to be the L^2 projections into V_h and Q_h respectively. Formally, for $\phi \in L^2(\Omega)$, $P_{V_h}\phi$ is the unique solution in V_h to

$$(P_{V_h}\phi - \phi, v_h) = 0 \quad \forall v_h \in V_h.$$

The projection operator P_{Q_h} is defined analogously. The treatment of the nonlinear terms is typically done through the definition of the skew-symmetric trilinear form,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) + \frac{1}{2}((\operatorname{div} \mathbf{u})\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})$$

which provides a more accurate energy balance than the usual convective formulation.

Suppressing a particular time discretization, which is not important at the moment, the semi-discrete Galerkin method is then to find $(\mathbf{u}_h(t), p_h(t)) \in (X_h, Q_h) \ \forall \ t > 0$ satisfying

$$((\mathbf{u}_h)_t, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = -(P_{Q_h}\varphi, \operatorname{div} \mathbf{v}_h) \quad (2.1)$$

$$(\operatorname{div} \mathbf{u}_h, q_h) = 0. \quad (2.2)$$

$\forall \ (\mathbf{v}_h, q_h) \in (X_h, Q_h)$ and $\forall \ t > 0$; $\mathbf{u}_h(0) = P_{V_h}(u_0)$. This scheme has been well-studied [9, 7], and it is known that it preserves a discrete analog of the NSE energy balance:

$$\text{NSE:} \quad \frac{1}{2}\|\mathbf{u}(T)\|^2 + \nu \int_0^T \|\nabla \mathbf{u}(t)\| \ dt = \frac{1}{2}\|\mathbf{u}(0)\|^2 \quad (2.3)$$

$$\text{Galerkin method:} \quad \frac{1}{2}\|\mathbf{u}_h(T)\|^2 + \nu \int_0^T \|\nabla \mathbf{u}_h(t)\| \ dt = \frac{1}{2}\|\mathbf{u}_h(0)\|^2 \quad (2.4)$$

However, no such discrete analog has been proven for helicity. The discussion below of the helicity balance of the Galerkin method brings us to the following conclusions:

- The usual Galerkin method (2.1)–(2.2) does not properly balance the discrete helicity $(\mathbf{u}_h, \operatorname{rot} \mathbf{u}_h)$, unless $\operatorname{rot} \mathbf{u}_h \in V_h$, which is a rare event for most practical choices of discrete spaces X_h and Q_h.
- The usual Galerkin method (2.1)–(2.2) better balances the modified discrete helicity $(\mathbf{u}_h, \mathbf{w}_h)$ where $\mathbf{w}_h$ is a solution of a discrete vorticity equation. Although, this balance is up to producing (numerical) helicity on boundaries.

2.1 Discrete helicity balance

Due to the definition of P_{V_h} and integration by parts it holds for sufficiently smooth $\mathbf{u}_h(t)$

$$((\mathbf{u}_h)_t, P_{V_h}\operatorname{rot} \mathbf{u}_h) = ((\mathbf{u}_h)_t, \operatorname{rot} \mathbf{u}_h) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{u}_h, \operatorname{rot} \mathbf{u}_h) \quad (2.5)$$

Testing (2.1) with $\mathbf{v}_h = 2P_{V_h}\operatorname{rot} \mathbf{u}_h$ and using (2.5) gives

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{u}_h, \operatorname{rot} \mathbf{u}_h) + 2b(\mathbf{u}_h, \mathbf{u}_h, P_{V_h}\operatorname{rot} \mathbf{u}_h) + 2\nu(\nabla \mathbf{u}_h, \nabla P_{V_h}\operatorname{rot} \mathbf{u}_h) = 0$$

Integrating in time leads to the following relation

$$H_G(T) + 2\nu \int_0^T (\nabla \mathbf{u}_h, \nabla P_{V_h}\operatorname{rot} \mathbf{u}_h) \ \mathrm{d}t + 2\int_0^T b(\mathbf{u}_h, \mathbf{u}_h, P_{V_h}\operatorname{rot} \mathbf{u}_h) \ \mathrm{d}t = H_G(0), \quad (2.6)$$

with $H_G(t) := (\mathbf{u}_h(t), \operatorname{rot} \mathbf{u}_h(t))$. Define the defect: $\boldsymbol{\xi}_h = \operatorname{rot} \mathbf{u}_h - P_{V_h}\operatorname{rot} \mathbf{u}_h$. The function $\boldsymbol{\xi}_h$ is non-zero since $\operatorname{rot} \mathbf{u}_h$ does not belong to the discrete velocity space. Moreover, $\boldsymbol{\xi}_h$ may experience a large variation near $\partial\Omega$ since V_h imposes no-slip boundary condition, which may not be satisfied by $\operatorname{rot}\mathbf{u}_h$. Recalling a vector identity and integrating by parts gives

$$(\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \operatorname{rot}\mathbf{u}_h) = (\operatorname{rot}\mathbf{u}_h\times\mathbf{u}_h, \operatorname{rot}\mathbf{u}_h) + \frac{1}{2}(\nabla|\mathbf{u}_h|^2, \operatorname{rot}\mathbf{u}_h) = \frac{1}{2}(\operatorname{rot}\nabla|\mathbf{u}_h|^2, \mathbf{u}_h) = 0,$$

which allows the helicity balance (2.6) to be written as

$$H_G(T) + 2\nu\int_0^T(\nabla\mathbf{u}_h, \nabla\operatorname{rot}\mathbf{u}_h)\,\mathrm{d}t = H_G(0) + \Phi(\boldsymbol{\xi}_h), \tag{2.7}$$

where

$$\begin{aligned}\Phi(\boldsymbol{\xi}_h) &= \int_0^T 2(\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \boldsymbol{\xi}_h) + 2\nu(\nabla\mathbf{u}_h, \nabla\boldsymbol{\xi}_h) + (\operatorname{div}\mathbf{u}_h, \mathbf{u}_h\cdot\operatorname{rot}\mathbf{u}_h) + ((\operatorname{div}\mathbf{u}_h)\mathbf{u}_h, \boldsymbol{\xi}_h)\mathrm{d}t \\ &= \int_0^T 2(\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \boldsymbol{\xi}_h) + 2\nu(\nabla\mathbf{u}_h, \nabla\boldsymbol{\xi}_h) + \inf_{q_h\in Q_h}(\operatorname{div}\mathbf{u}_h, \mathbf{u}_h\cdot\operatorname{rot}\mathbf{u}_h - q_h) \\ &\quad + \inf_{q_h\in Q_h}(\operatorname{div}\mathbf{u}_h, \mathbf{u}_h\cdot\boldsymbol{\xi}_h - q_h)\mathrm{d}t.\end{aligned}$$

If the convective approximation of the nonlinear term is used, i.e. $b(\mathbf{u},\mathbf{v},\mathbf{w}) := (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})$, then the extra terms are reduced to

$$\Phi(\boldsymbol{\xi}_h) = 2\int_0^T(\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \boldsymbol{\xi}_h) + \nu(\nabla\mathbf{u}_h, \nabla\boldsymbol{\xi}_h)\,\mathrm{d}t.$$

However in the latter case the discrete energy balance (2.4) is altered.

In either case the relation (2.7) does not act as a discrete analog of the continuous NSE's helicity balance (1.9), because of the extra term $\Phi(\boldsymbol{\xi}_h)$. Specifically, in (2.7) the term $\Phi(\boldsymbol{\xi}_h)$ doesn't necessarily vanish, unless $\operatorname{rot}\mathbf{u}_h \in V_h$ (and $\operatorname{div}\mathbf{u}_h = 0$ for skew-symmetric convection approximation), which is a rare event for most practical choices of discrete spaces X_h and Q_h; in particular, the nonlinear term in $\Phi(\boldsymbol{\xi}_h)$ cannot be proven small without an unrealistic small data condition. Thus the behavior of this term is unknown, as is its possibly nonphysical effect. The extra dissipation term $2\nu\int_0^T(\nabla\mathbf{u}_h, \nabla\boldsymbol{\xi}_h)$ may create an error due to a large variation of $\boldsymbol{\xi}_h$ near boundary. The last two entries of $\Phi(\boldsymbol{\xi}_h)$ for the skew-symmetric approximation does not vanish with the exception of special choice of X_h and Q_h (for example the Scott-Vogelius finite elements), which enforce the discrete Galerkin solution to be strongly divergence-free.

2.2 Modified discrete helicity balance

The apparent lack of a physically accurate discrete helicity balance for the Galerkin discretization of the 3D NSE suggests a drawback to the scheme. However, by slightly altering the discrete definition of helicity, we find that the method indeed admits an accurate balance. If the vorticity used is instead the solution of a particular vorticity equation, then the physical balance is recovered up to the boundary effect. This alternative balance does not change the Galerkin scheme, since the vorticity equation is computed a posteriori.

The discrete counterpart of the vorticity equation (1.4) is defined by applying the Galerkin method to (1.4) subject to appropriate boundary and initial conditions. The discrete vorticity boundary condition needs to be different than that for velocity because it is not physical that vorticity satisfy homogeneous Dirichlet boundary conditions when the velocity does. Since the system (2.1)-(2.2) does not include vorticity, we can take its velocity solution $\mathbf{u}_h(t)$ as input to a vorticity equation. An appropriate definition for the vorticity space $\widetilde{X}_h$ would be the same elements as is chosen for X_h (which we have left general), but without enforcing homogeneous Dirichlet boundary condition. This leads us to the discrete vorticity equation: For given $\mathbf{u}_h(t)$, find $(\mathbf{w}_h(t), \lambda_h(t)) \in (\widetilde{X}_h, Q_h)$

satisfying $\forall\ (\mathbf{v}_h, q_h) \in (X_h, Q_h)$ and $\forall\ t > 0$

$$((\mathbf{w}_h)_t, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h) - b(\mathbf{w}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) + (\lambda_h, \operatorname{div} \mathbf{v}_h) = 0, \tag{2.8}$$

$$(\operatorname{div} \mathbf{w}_h, q_h) = 0, \tag{2.9}$$

$$\mathbf{w}_h = I_h(\operatorname{rot} \mathbf{u}_h) \quad \text{on} \quad \partial\Omega \tag{2.10}$$

$$\mathbf{w}_h = I_h(\operatorname{rot} \mathbf{u}_0) \quad \text{for} \quad t = 0, \tag{2.11}$$

here I_h is an interpolation operator from rot (X_h) to $\widetilde{X}_h$, and λ_h is a formal Lagrange multiplier corresponding to the discrete divergence-free condition for vorticity. Apart of the boundary condition, $\mathbf{w}_h$ belongs to the same discrete space as velocity and due to (2.9), satisfies the discrete divergence-free constraint imposed for functions in V_h, and due to properties of L^2-orthogonal projection, the defect $\boldsymbol{\xi}_h = (I - P_{V_h})\mathbf{w}_h$ satisfies

$$\|\boldsymbol{\xi}_h\| = \inf_{\mathbf{v}_h \in V_h} \|\mathbf{w}_h - \mathbf{v}_h\| \,.$$

For the Galerkin discretization of the 3D NSE, denote by $H_h(t)$ the alternative discrete helicity at time t:

$$H_h(t) := \int_\Omega \mathbf{u}_h(t) \cdot \mathbf{w}_h(t) \, \mathrm{d}\mathbf{x},$$

where $\mathbf{u}_h(t)$ is the velocity solution of (2.1)–(2.2) and $\mathbf{w}_h(t)$ is the vorticity solution of (2.8)–(2.11). Using this definition of discrete helicity we prove the following result.

Theorem 2.1. *The solution of (2.1)-(2.2) satisfies the following discrete helicity balance.*

$$H_h(T) + 2\nu \int_0^T (\nabla \mathbf{u}_h, \nabla \mathbf{w}_h) \mathrm{d}t = H_h(0) + \Phi(\boldsymbol{\xi}_h), \tag{2.12}$$

where

$$\Phi(\boldsymbol{\xi}_h) = \int_0^T b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\xi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\xi}_h) \ \mathrm{d}t.$$

Proof. Choose $\mathbf{v}_h = P_{V_h}\mathbf{w}_h$ in (2.1) and $\mathbf{v}_h = \mathbf{u}_h$ in (2.8), and since $P_{V_h}\mathbf{w}_h$ and $\mathbf{u}_h$ are in V_h, this vanishes the pressure terms and one of the trilinear terms in (2.8), yielding

$$\begin{aligned} ((\mathbf{u}_h)_t, \mathbf{w}_h) + b(\mathbf{u}_h, \mathbf{u}_h, P_{V_h}\mathbf{w}_h) + \nu(\nabla \mathbf{u}_h, \nabla P_{V_h}\mathbf{w}_h) &= 0, \\ ((\mathbf{w}_h)_t, \mathbf{u}_h) + b(\mathbf{u}_h, \mathbf{w}_h, \mathbf{u}_h) + \nu(\nabla \mathbf{w}_h, \nabla \mathbf{u}_h) &= 0. \end{aligned}$$

Note that by definition the trilinear term changes sign if we switch the order of the second and third arguments. Thus adding the equations leaves only trilinear term with $\boldsymbol{\xi}_h = \mathbf{w}_h - P_{V_h}\mathbf{w}_h$:

$$((\mathbf{u}_h)_t, \mathbf{w}_h) + ((\mathbf{w}_h)_t, \mathbf{u}_h) - b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\xi}_h) + 2\nu(\nabla \mathbf{u}_h, \nabla \mathbf{w}_h) - \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\xi}_h) = 0.$$

This can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{u}_h, \mathbf{w}_h) - b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\xi}_h) + 2\nu(\nabla \mathbf{u}_h, \nabla \mathbf{w}_h) - \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\xi}_h) = 0.$$

Integrating over time for $t \in [0, T]$ yields (2.12). □

The relation (2.12) can be observed as a consistent discrete counterpart of the NSE helicity balance (1.9) if $\Phi(\boldsymbol{\xi}_h)$ on the righthand side of (2.12) vanishes. This is the case for a periodic problem, as stated in the following corollary.

Corollary 2.1. *If (2.1)-(2.2) is equipped with periodic boundary conditions, then its solutions satisfy the following exact discrete helicity balance.*

$$H_h(T) + 2\nu \int_0^T (\nabla \mathbf{u}_h, \nabla \mathbf{w}_h) \mathrm{d}t = H_h(0). \tag{2.13}$$

Proof. This follows from Theorem 2.1 since we can take the vorticity space $\widetilde{X}_h = X_h$, which implies $\boldsymbol{\xi}_h = 0$ and thus $\Phi(\boldsymbol{\xi}_h) = 0$. □

For more practical boundary conditions, however, $\Phi(\boldsymbol{\xi}_h)$ does not necessarily vanish; the gradient of the driving function $\boldsymbol{\xi}_h$ can be large in a near-boundary. Thus the discrete helicity balance (2.12) can be altered (only) by boundary effects.

Remark 2.1. While the results in this note were shown for the semi-discrete method, an extension to many particular time-discretizations, e.g. Crank-Nicolson, semi-implicit or Chorin-Temam splitting methods, is an easy exercise. One should be careful to make sure the time-discretization of the vorticity equation is consistent with the one chosen for the velocity equation, and perform a summation over discrete times instead of time integration.

3 Discussion

One important goal for the design of numerical schemes is to match as much of the problems true conservation laws as possible, while reducing the problem to one that can be efficiently computed. Beginning with Arakawa and his 2D energy and enstrophy preserving scheme for the NSE [1], it has been observed that sometimes small modifications of known methods increase physical accuracy without a significant increase in computational work. For the shallow water equations, alterations of energy conserving schemes became energy and potential enstrophy conserving schemes [2, 5]. For the 3D NSE, energy and helicity conserving schemes were developed to mirror the analogous conservation laws; the scheme of [10] is restricted to axisymmetric flow, and that of [14] is for the full 3D NSE but in the perioidic setting. To our knowledge, no numerical scheme exists that correctly matches the discrete physical behavior of energy and helicity to that of the continuous 3D NSE for homogeneous Dirichlet boundary conditions, the simplest relevant case.

This work shows that, in the periodic setting, the usual Galerkin method with explicitly skew-symmetrized nonlinear term accurately balances both energy and a discrete helicity. Unfortunately, when we look to extend this result to homogeneous Dirichlet boundary conditions, we find the scheme may generate numerical helicity near the boundary. The complete understanding of the case of non-periodic boundary conditions is an important open problem.

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