6.2: Solution of initial value problems

Topics:

- Properties of Laplace transform, with proofs and examples
- Inverse Laplace transform, with examples, review of partial fraction,
- Solution of initial value problems, with examples covering various cases.

Properties of Laplace transform:

1. Linearity: \( \mathcal{L}\{c_1f(t) + c_2g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\} \).

2. First derivative: \( \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \).

3. Second derivative: \( \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \).

4. Higher order derivative:

\[
\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).
\]

5. \( \mathcal{L}\{-tf(t)\} = F'(s) \) where \( F(s) = \mathcal{L}\{f(t)\} \). This also implies \( \mathcal{L}\{tf(t)\} = -F'(s) \).

6. \( \mathcal{L}\{e^{at}f(t)\} = F(s-a) \) where \( F(s) = \mathcal{L}\{f(t)\} \). This implies \( e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\} \).

Remarks:

- Note property 2 and 3 are useful in differential equations. It shows that each derivative in \( t \) caused a multiplication of \( s \) in the Laplace transform.

- Property 5 is the counter part for Property 2. It shows that each derivative in \( s \) causes a multiplication of \( -t \) in the inverse Laplace transform.

- Property 6 is also known as the Shift Theorem. A counter part of it will come later in chapter 6.3.
**Proof:**

1. This follows by definition.

2. By definition

\[ \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt \bigg|_0^\infty - \int_0^\infty (-s)e^{-st}f(t)dt = -f(0) + s\mathcal{L}\{f(t)\}. \]

3. This one follows from Property 2. Set \( f \) to be \( f' \) we get

\[ \mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \]

4. This follows by induction, using property 2.

5. The proof follows from the definition:

\[ F'(s) = \frac{d}{ds} \int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)dt = \int_0^\infty (-t)e^{-st}f(t)dt = \mathcal{L}\{-tf(t)\}. \]

6. This proof also follows from definition:

\[ \mathcal{L}\{e^{at}f(t)\} \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a). \]

By using these properties, we could find more easily Laplace transforms of many other functions.

**Example 1.**

From \( \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \) we get \( \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}. \)

**Example 2.**

From \( \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}, \) we get \( \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}. \)
Example 3.

From \( \mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2} \), we get \( \mathcal{L}\{e^{at}\cos bt\} = \frac{s - a}{(s - a)^2 + b^2} \).

Example 4.

\[
\mathcal{L}\{t^3 + 5t - 2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + \frac{5}{s^2} - 2\frac{1}{s}.
\]

Example 5.

\[
\mathcal{L}\{e^{2t}(t^3 + 5t - 2)\} = \frac{3!}{(s - 2)^4} + \frac{5}{(s - 2)^2} - 2\frac{1}{s - 2}.
\]

Example 6.

\[
\mathcal{L}\{(t^2 + 4)e^{2t} - e^{-t}\cos t\} = \frac{2}{(s - 2)^3} + \frac{4}{s - 2} - \frac{s + 1}{(s + 1)^2 + 1},
\]

because

\[
\mathcal{L}\{t^2 + 4\} = \frac{2}{s^3} + \frac{4}{s}, \quad \Rightarrow \quad \mathcal{L}\{(t^2 + 4)e^{2t}\} = \frac{2}{(s - 2)^3} + \frac{4}{s - 2}.
\]

Next are a few examples for Property 5.

Example 7.

Given \( \mathcal{L}\{e^{at}\} = \frac{1}{s - a} \), we get \( \mathcal{L}\{te^{at}\} = -\left(\frac{1}{s - a}\right)' = \frac{1}{(s - a)^2} \).

Example 8.

\[
\mathcal{L}\{t\sin bt\} = -\left(\frac{b}{s^2 + b^2}\right)' = \frac{-2bs}{(s^2 + b^2)^2}.
\]

Example 9.

\[
\mathcal{L}\{t\cos bt\} = -\left(\frac{s}{s^2 + b^2}\right)' = \cdots = \frac{s^2 - b^2}{(s^2 + b^2)^2}.
\]
Inverse Laplace transform. Definition:

\[ \mathcal{L}^{-1}\{F(s)\} = f(t), \quad \text{if} \quad F(s) = \mathcal{L}\{f(t)\}. \]

Technique: find the way back.

Some simple examples:

**Example 10.**

\[ \mathcal{L}^{-1}\left\{ \frac{3}{s^2 + 4} \right\} = \mathcal{L}^{-1}\left\{ \frac{3}{2} \cdot \frac{2}{s^2 + 2^2} \right\} = \frac{3}{2} \mathcal{L}^{-1}\left\{ \frac{2}{s^2 + 2^2} \right\} = \frac{3}{2} \sin 2t. \]

**Example 11.**

\[ \mathcal{L}^{-1}\left\{ \frac{2}{(s + 5)^4} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{3} \cdot \frac{6}{(s + 5)^4} \right\} = \frac{1}{3} \mathcal{L}^{-1}\left\{ \frac{3!}{(s + 5)^4} \right\} = \frac{1}{3} e^{-5t} \mathcal{L}^{-1}\left\{ \frac{3!}{s^4} \right\} = \frac{1}{3} e^{-5t} t^3. \]

**Example 12.**

\[ \mathcal{L}^{-1}\left\{ \frac{s + 1}{s^2 + 4} \right\} = \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 4} \right\} + \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{2}{s^2 + 4} \right\} = \cos 2t \frac{1}{2} \sin 2t. \]

**Example 13.**

\[ \mathcal{L}^{-1}\left\{ \frac{s + 1}{s^2 - 4} \right\} = \mathcal{L}^{-1}\left\{ \frac{s + 1}{(s - 2)(s + 2)} \right\} = \mathcal{L}^{-1}\left\{ \frac{3/4}{s - 2} + \frac{1/4}{s + 2} \right\} = \frac{3}{4} e^{2t} + \frac{1}{4} e^{-2t}. \]

Here we used [partial fraction to find out:](text)

\[ \frac{s + 1}{(s - 2)(s + 2)} = \frac{A}{s - 2} + \frac{B}{s + 2}, \quad A = \frac{3}{4}, \quad B = \frac{1}{4}. \]
Solutions of initial value problems.
We will go through one example first.

**Example 14.** (Two distinct real roots.) Solve the initial value problem by Laplace transform,

\[ y'' - 3y' - 10y = 2, \quad y(0) = 1, \, y'(0) = 2. \]

**Answer.** Step 1. Take Laplace transform on both sides: Let \( \mathcal{L}\{y(t)\} = Y(s) \), and then

\[
\begin{align*}
\mathcal{L}\{y'(t)\} &= sY(s) - y(0) = sY - 1, \\
\mathcal{L}\{y''(t)\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y - s - 2.
\end{align*}
\]

Note the initial conditions are the first thing to go in!

\[
\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}, \quad \Rightarrow \quad s^2Y - s - 2 - 3(sY - 1) - 10Y = \frac{2}{s}.
\]

Now we get an algebraic equation for \( Y(s) \).

Step 2: Solve it for \( Y(s) \):

\[
(s^2 - 3s - 10)Y(s) = \frac{2}{s} + s + 2 - 3 = \frac{s^2 - s + 2}{s} , \quad \Rightarrow \quad Y(s) = \frac{s^2 - s + 2}{s(s - 5)(s + 2)}.
\]

Step 3: Take inverse Laplace transform to get \( y(t) = \mathcal{L}^{-1}\{Y(s)\} \). The main technique here is **partial fraction**.

\[
Y(s) = \frac{s^2 - s + 2}{s(s - 5)(s + 2)} = \frac{A}{s} + \frac{B}{s - 5} + \frac{C}{s + 2} = \frac{A(s - 5)(s + 2) + Bs(s + 2) + Cs(s - 5)}{s(s - 5)(s + 2)}.
\]

Compare the numerators:

\[
s^2 - s + 2 = A(s - 5)(s + 2) + Bs(s + 2) + Cs(s - 5).
\]

The previous equation holds for all values of \( s \).

Set \( s = 0 \): we get \(-10A = 2\), so \( A = -\frac{1}{5} \).

Set \( s = 5 \): we get \( 35B = 22\), so \( B = \frac{22}{35} \).

Set \( s = -2 \): we get \( 14C = 8\), so \( C = \frac{4}{7} \).

Now, \( Y(s) \) is written into sum of terms which we can find the inverse transform:

\[
y(t) = A\mathcal{L}^{-1}\{\frac{1}{s}\} + B\mathcal{L}^{-1}\{\frac{1}{s - 5}\} + C\mathcal{L}^{-1}\{\frac{1}{s + 2}\} = -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}.
\]
Structure of solutions:

- Take Laplace transform on both sides. You will get an algebraic equation for $Y$.
- Solve this equation to get $Y(s)$.
- Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y\}$.

**Example 15.** (Distinct real roots, but one matches the source term.) Solve the initial value problem by Laplace transform,

$$y'' - y' - 2y = e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$  

**Answer.** Take Laplace transform on both sides of the equation, we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}, \quad \Rightarrow \quad s^2Y(s) - sY(s) - 2Y(s) = \frac{1}{s - 2}.$$  

Solve it for $Y$:

$$(s^2 - s - 2)Y(s) = \frac{1}{s - 2} + 1 = \frac{s - 1}{s - 2}, \quad \Rightarrow \quad Y(s) = \frac{s - 1}{(s - 2)(s^2 - s - 2)} = \frac{s - 1}{(s - 2)^2(s + 1)}.$$  

Use partial fraction:

$$\frac{s - 1}{(s - 2)^2(s + 1)} = \frac{A}{s + 1} + \frac{B}{s - 2} + \frac{C}{(s - 2)^2}.$$  

Compare the numerators:

$$s - 1 = A(s - 2)^2 + B(s + 1)(s - 2) + C(s + 1)$$

Set $s = -1$, we get $A = -\frac{2}{9}$.

Set $s = 2$, we get $C = \frac{1}{3}$.

Set $s = 0$ (any convenient values of $s$ can be used in this step), we get $B = \frac{2}{9}$.

So

$$Y(s) = -\frac{2}{9}\frac{1}{s + 1} + 2\frac{1}{9}\frac{1}{s - 2} + \frac{1}{3}\frac{1}{(s - 2)^2}$$

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\[ y(t) = \mathcal{L}^{-1}\{Y\} = -\frac{2}{9}e^{-t} + \frac{2}{9}e^{2t} + \frac{1}{3}te^{2t}. \]

Compare this to the method of undetermined coefficients: general solution of the equation should be \( y = y_H + Y \), where \( y_H \) is the general solution to the homogeneous equation and \( Y \) is a particular solution. The characteristic equation is \( r^2 - r - 2 = (r + 1)(r - 2) = 0 \), so \( r_1 = -1, r_2 = 2 \), and \( y_H = c_1e^{-t} + c_2e^{2t} \). Since 2 is a root, so the form of the particular solution is \( Y = At e^{2t} \). This discussion concludes that the solution should be of the form

\[ y = c_1e^{-t} + c_2e^{2t} + At e^{2t} \]

for some constants \( c_1, c_2, A \). This fits well with our result.

**Example 16. (Complex roots.)** Solve

\[ y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1. \]

**Answer.** Before we solve it, let’s use the method of undetermined coefficients to find out which terms will be in the solution.

\[ r^2 - 2r + 2 = 0, \quad (r - 1)^2 + 1 = 0, \quad r_{1,2} = 1 \pm i, \]

\[ y_H = c_1e^t \cos t + c_2e^t \sin t, \quad Y = Ae^{-t}, \]

so the solution should have the form:

\[ y = y_H + Y = c_1e^t \cos t + c_2e^t \sin t + Ae^{-t}. \]

The Laplace transform would be

\[ Y(s) = c_1 \frac{s - 1}{(s - 1)^2 + 1} + c_2 \frac{1}{(s - 1)^2 + 1} + A \frac{1}{s + 1} = \frac{c_1(s - 1) + c_2}{(s - 1)^2 + 1} + \frac{A}{s + 1}. \]

This gives us some idea on which terms to look for in partial fraction.

Now let’s use the Laplace transform:

\[ Y(s) = \mathcal{L}\{y\}, \quad \mathcal{L}\{y'\} = sY - y(0) = sY, \]

\[ \mathcal{L}\{y''\} = s^2Y - sy(0) - y(0) = s^2Y - 1. \]
\[ s^2 Y - 1 - 2sY + 2Y = \frac{1}{s + 1}, \quad \Rightarrow \quad (s^2 - 2s + 2)Y(s) = \frac{1}{s + 1} + 1 = \frac{s + 2}{s + 1} \]

\[ Y(s) = \frac{s + 2}{(s + 1)(s^2 - 2s + 2)} = \frac{s + 2}{(s + 1)((s - 1)^2 + 1)} = \frac{A}{s + 1} + \frac{B(s - 1) + C}{(s - 1)^2 + 1} \]

Compare the numerators:

\[ s + 2 = A((s - 1)^2 + 1) + (B(s - 1) + C)(s + 1). \]

Set \( s = -1 \): \( 5A = 1, A = \frac{1}{5}. \)

Compare coefficients of \( s^2 \)-term: \( A + B = 0, B = -A = -\frac{1}{5}. \)

Set any value of \( s \), say \( s = 0 \): \( 2 = 2A - B + C, C = 2 - 2A + B = \frac{9}{5}. \)

\[ Y(s) = \frac{1}{5} \frac{1}{s + 1} - \frac{1}{5} \frac{s - 1}{(s - 1)^2 + 1} + \frac{9}{5} \frac{1}{(s - 1)^2 + 1} \]

\[ y(t) = \frac{1}{5} e^{-t} - \frac{1}{5} e^t \cos t + \frac{9}{5} e^t \sin t. \]

We see this fits our prediction.

**Example 17.** (Pure imaginary roots.) Solve

\[ y'' + y = \cos 2t, \quad y(0) = 2, \quad y'(0) = 1. \]

**Answer.** Again, let’s first predict the terms in the solution:

\[ r^2 + 1 = 0, \quad r_{1,2} = \pm i, \quad y_H = c_1 \cos t + c_2 \sin t, \quad Y = A \cos 2t \]

so

\[ y = y_H + Y = c_1 \cos t + c_2 \sin t + A \cos 2t, \]

and the Laplace transform would be

\[ Y(s) = c_1 \frac{s}{s^2 + 1} + c_2 \frac{1}{s^2 + 1} + A \frac{s}{s^2 + 4}. \]

Now, let’s take Laplace transform on both sides:

\[ s^2 Y - 2s - 1 + Y = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4} \]
\[ (s^2 + 1)Y(s) = \frac{s}{s^2 + 4} + 2s + 1 = \frac{2s^3 + s^2 + 9s + 4}{s^2 + 4} \]

\[ Y(s) = \frac{2s^3 + s^2 + 9s + 4}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}. \]

Comparing numerators, we get

\[ 2s^3 + s^2 + 9s + 4 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1). \]

One may expand the right-hand side and compare terms to find \(A, B, C, D,\) but that takes more work.

Let’s try by setting \(s\) into complex numbers.

Set \(s = i,\) and remember the facts \(i^2 = -1\) and \(i^3 = -i,\) we have

\[-2i - 1 + 9i + 4 = (Ai + B)(-1 + 4),\]

which gives

\[3 + 7i = 3B + 3Ai, \quad \Rightarrow \quad B = 1, \quad A = \frac{7}{3}.\]

Set now \(s = 2i:\)

\[-16i - 4 + 18i + 4 = (2Ci + D)(-3),\]

then

\[0 + 2i = -3D - 6Ci, \quad \Rightarrow \quad D = 0, \quad C = -\frac{1}{3}.\]

So

\[ Y(s) = \frac{7}{3} \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{1}{3} \frac{s}{s^2 + 4} \]

and

\[ y(t) = \frac{7}{3} \cos t + \sin t - \frac{1}{3} \cos 2t. \]

A very brief review on partial fraction, targeted towards inverse Laplace transform.

Goal: rewrite a fractional form \(\frac{P_n(s)}{P_m(s)}\) (where \(P_n\) is a polynomial of degree \(n\)) into sum of “simpler” terms. We assume \(n < m.\)
The type of terms appeared in the partial fraction is solely determined by the denominator \( P_m(s) \). First, fact out \( P_m(s) \), write it into product of terms of (i) \( s - a \), (ii) \( s^2 + a^2 \), (iii) \((s_0)^2 + b^2\). The following table gives the terms in the partial fraction and their corresponding inverse Laplace transform.

<table>
<thead>
<tr>
<th>term in ( P_m(s) )</th>
<th>from where?</th>
<th>term in partial fraction</th>
<th>inverse L.T.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s - a )</td>
<td>real root, or ( g(t) = e^{at} )</td>
<td>( \frac{A}{s - a} )</td>
<td>( Ae^{at} )</td>
</tr>
<tr>
<td>( (s - a)^2 )</td>
<td>double roots, or ( r = a ) and ( g(t) = e^{at} )</td>
<td>( \frac{A}{s - a} + \frac{B}{(s - a)^2} )</td>
<td>( Ae^{at} + Bte^{at} )</td>
</tr>
<tr>
<td>( (s - a)^3 )</td>
<td>double roots, ( g(t) = e^{at} )</td>
<td>( \frac{A}{s - a} + \frac{B}{(s - a)^2} + \frac{C}{(s - a)^3} )</td>
<td>( Ae^{at} + Bte^{at} + \frac{Ct^2e^{at}}{2} )</td>
</tr>
<tr>
<td>( s^2 + \mu^2 )</td>
<td>imaginary roots or ( g(t) = \cos \mu t ) or ( \sin \mu t )</td>
<td>( \frac{As + B}{s^2 + \mu^2} )</td>
<td>( A \cos \mu t + B \sin \mu t )</td>
</tr>
<tr>
<td>( (s - \lambda)^2 + \mu^2 )</td>
<td>complex roots, ( g(t) = e^{\lambda t} \cos \mu t ) or ( \sin \mu t )</td>
<td>( \frac{A(s - \lambda) + B}{(s - \lambda)^2 + \mu^2} )</td>
<td>( e^{\lambda t}(A \cos \mu t + B \sin \mu t) )</td>
</tr>
</tbody>
</table>

In summary, this table can be written

\[
P_m(s) = \frac{A}{s - a} + \frac{B_1}{s - b} + \frac{B_2}{(s - b)^2} + \frac{C_1}{s - c} + \frac{C_2}{(s - c)^2} + \frac{C_3}{(s - c)^3} + \frac{D_1(s - \lambda) + D_2}{(s - \lambda)^2 + \mu^2}.
\]