Numerical Linear Algebra
and
Image Restoration

Maui High Performance Computing Center
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Review from Wednesday

- **Problem:** $\mathrm{b} = A\mathrm{x} + \mathrm{e}$

- **Solution:** Compute a filtered solution:

$$\mathbf{x}_{\text{filt}} = \sum_{i=1}^{n} \phi \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

- **Efficiency:** Exploiting matrix structure, can use:

  $A = V^* \Lambda V$ for circulant

  $A = V^* \Lambda V$ for symmetric Toeplitz + Hankel

  $A = U \Sigma V^T$ for Kronecker products

Thursday Topics

1. Defining A from PSF and boundary conditions.
2. Kronecker product decomposition of A
   (even if PSF is not separable)
3. SVD approximations

Defining A from PSF

- Using linear algebra notation, the i-th column of A can be written as:

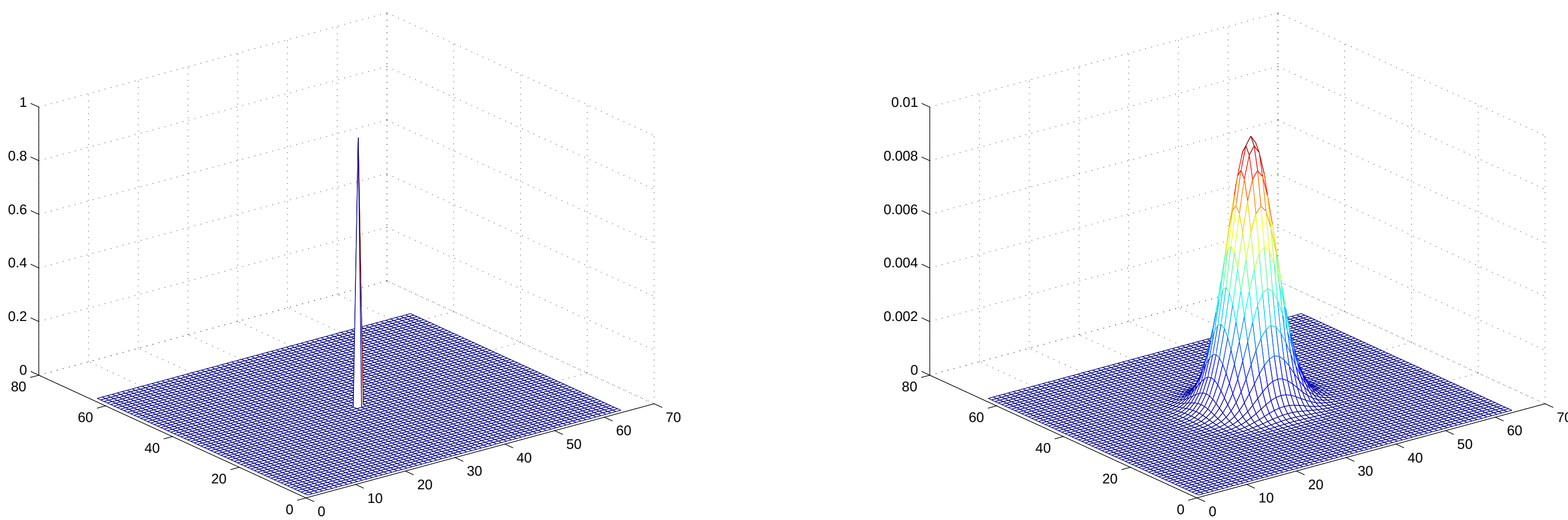
$$A\mathbf{e}_i = \left[\begin{array}{ccccc} \mathbf{a}_1 & \cdots & \mathbf{a}_i & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array}\right] = \mathbf{a}_i$$

- In an imaging system,

$$\begin{aligned} \mathbf{e}_i &= \text{point source} \\ A\mathbf{e}_i &= \text{point spread function (PSF)} \end{aligned}$$

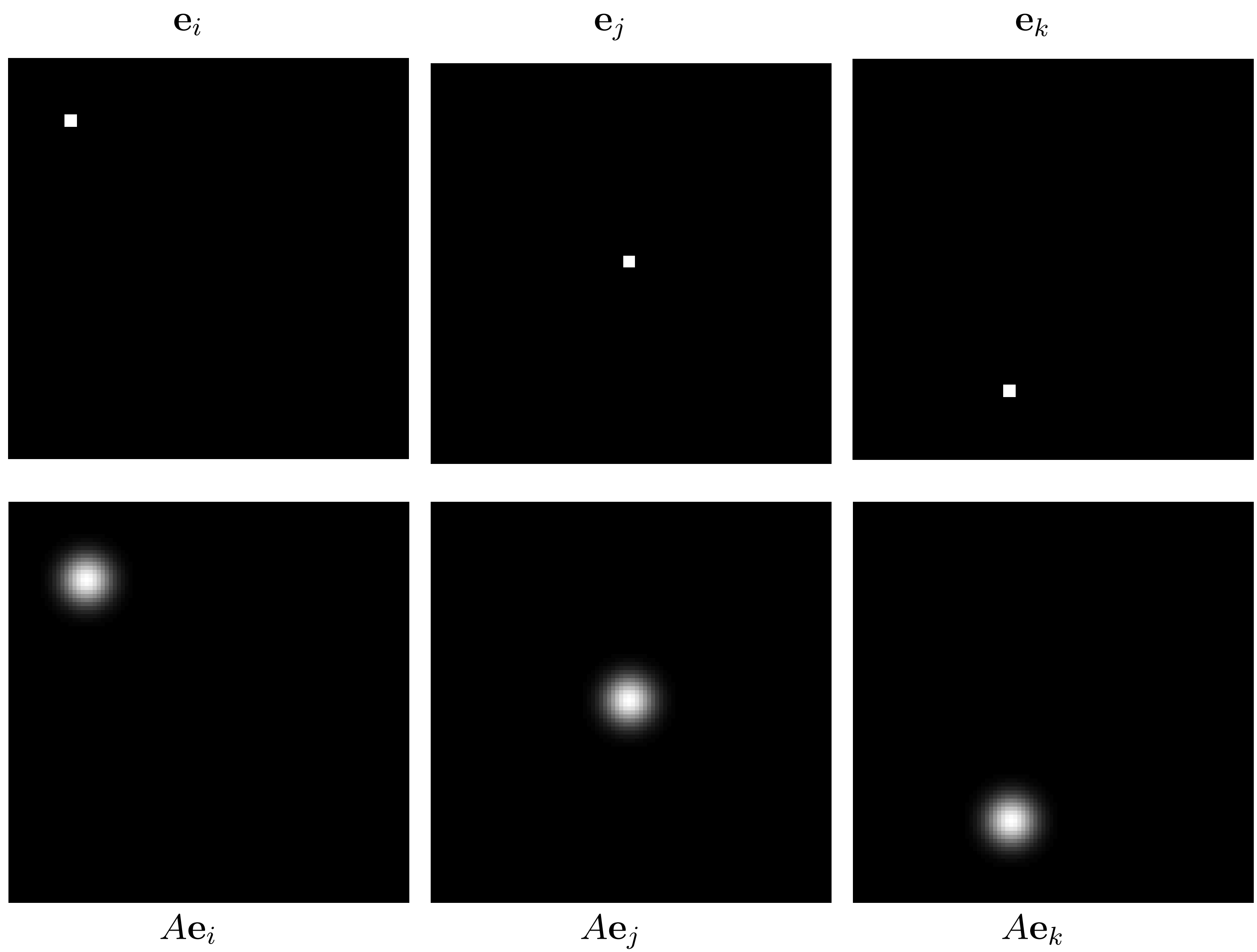
Defining A from PSF

point source PSF



Defining A from PSF

Spatially invariant PSF implies:



Defining A from PSF

That is, spatially invariant implies

- Each column of A is identical, modulo shift.
- One point PSF is enough to fully describe A.
- A has Toeplitz structure.

Defining A from PSF

$$\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{ blur } \rightarrow \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \rightarrow A\mathbf{e}_5 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c} & p_{11} & \\ & p_{12} & \\ & p_{13} & \\ \hline & p_{21} & \\ & p_{22} & \\ & p_{23} & \\ \hline & p_{31} & \\ & p_{32} & \\ & p_{33} & \end{array}\right]$$

Defining A from PSF

$$\mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{blur} \rightarrow \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \rightarrow A\mathbf{e}_5 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|ccc|ccc} p_{22} & p_{21} & & p_{12} & p_{11} & & & & \\ p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} & & & \\ & p_{23} & p_{22} & & p_{13} & p_{12} & & & \\ \hline p_{32} & p_{31} & & p_{22} & p_{21} & & p_{12} & p_{11} & \\ p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} \\ & p_{33} & p_{32} & & p_{23} & p_{22} & & p_{13} & p_{12} \\ \hline & & & p_{32} & p_{31} & & p_{22} & p_{21} & \\ & & & p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} \\ & & & & p_{33} & p_{32} & & p_{23} & p_{22} \end{array}\right]$$

Defining A from PSF

Including the boundary condition:

zero

$$A = \left[\begin{array}{ccc|ccc|ccc} p_{22} & p_{21} & 0 & p_{12} & p_{11} & 0 & 0 & 0 & 0 \\ p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & p_{23} & p_{22} & 0 & p_{13} & p_{12} & 0 & 0 & 0 \\ \hline p_{32} & p_{31} & 0 & p_{22} & p_{21} & 0 & p_{12} & p_{11} & 0 \\ p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} \\ 0 & p_{33} & p_{32} & 0 & p_{23} & p_{22} & 0 & p_{13} & p_{12} \\ \hline 0 & 0 & 0 & p_{32} & p_{31} & 0 & p_{22} & p_{21} & 0 \\ 0 & 0 & 0 & p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} \\ 0 & 0 & 0 & 0 & p_{33} & p_{32} & 0 & p_{23} & p_{22} \end{array}\right]$$

BTTB

Defining A from PSF

Including the boundary condition:

periodic

$$A = \left[\begin{array}{ccc|ccc|ccc} p_{22} & p_{21} & p_{23} & p_{12} & p_{11} & p_{13} & p_{32} & p_{31} & p_{33} \\ p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} & p_{33} & p_{32} & p_{31} \\ p_{21} & p_{23} & p_{22} & p_{11} & p_{13} & p_{12} & p_{31} & p_{33} & p_{32} \\ \hline p_{32} & p_{31} & p_{33} & p_{22} & p_{21} & p_{23} & p_{12} & p_{11} & p_{13} \\ p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} & p_{13} & p_{12} & p_{11} \\ p_{31} & p_{33} & p_{32} & p_{21} & p_{23} & p_{22} & p_{11} & p_{13} & p_{12} \\ \hline p_{12} & p_{11} & p_{13} & p_{32} & p_{31} & p_{33} & p_{22} & p_{21} & p_{23} \\ p_{13} & p_{12} & p_{11} & p_{33} & p_{32} & p_{31} & p_{23} & p_{22} & p_{21} \\ p_{11} & p_{13} & p_{12} & p_{31} & p_{33} & p_{32} & p_{21} & p_{23} & p_{22} \end{array}\right]$$

BCCB = BTTB + BTTB

Defining A from PSF

Including the boundary condition:

reflexive

$$A = \text{BTTB} + \text{BTHB} + \text{BHTB} + \text{BHHB}$$

Defining A from PSF

For a separable PSF, we get:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \mathbf{c}\mathbf{d}^T = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 \\ c_2d_1 & c_2d_2 & c_2d_3 \\ c_3d_1 & c_3d_2 & c_3d_3 \end{bmatrix} \rightarrow A\mathbf{e}_5 = \begin{bmatrix} c_1 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ c_2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ c_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \end{bmatrix}$$

$$\left[\begin{array}{c|c|c} & c_1 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} & \\ \hline & c_2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} & \\ \hline & c_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} & \end{array}\right]$$

Defining A from PSF

For a separable PSF, we get:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \mathbf{c}\mathbf{d}^T = \begin{bmatrix} c_1 d_1 & c_1 d_2 & c_1 d_3 \\ c_2 d_1 & c_2 d_2 & c_2 d_3 \\ c_3 d_1 & c_3 d_2 & c_3 d_3 \end{bmatrix} \rightarrow A\mathbf{e}_5 = \begin{bmatrix} c_1 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ c_2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ c_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \end{bmatrix}$$

$$\left[\begin{array}{c|c|c} c_2 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} & c_1 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} & \\ \hline c_3 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} & c_2 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} & c_1 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} \\ \hline & c_3 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} & c_2 \begin{pmatrix} d_2 & d_1 & \\ d_3 & d_2 & d_1 \\ & d_3 & d_2 \end{pmatrix} \end{array}\right] = C \otimes D$$

Defining A from PSF

Kronecker Product Summary:

If point spread function satisfies: $P = \mathbf{c}\mathbf{d}^T$, then

- Zero BC $\Rightarrow A = C \otimes D$, where C and D are Toeplitz
- Periodic BC $\Rightarrow A = C \otimes D$, where C and D are circulant
- Reflexive BC $\Rightarrow A = C \otimes D$, where C and D are Toeplitz+Hankel

Defining A from PSF

Matrix Summary

BC	PSF Not Separable	PSF Separable
zero	BTTB	Toep$\otimes$Toep
periodic	BCCB	Circ$\otimes$Circ
reflexive	BTTB+BTHB +BHTB+BHHB	(Toep+Hank)$\otimes$(Toep+Hank)
reflexive with symm. PSF	same as reflexive with symmetry	same as reflexive with symmetry

Defining A from PSF

Matrix Summary

BC	PSF Not Separable	PSF Separable
zero	BTTB	use SVD
periodic	use FFT	use FFT
reflexive	BTTB+BTHB +BHTB+BHHB	use SVD
reflexive with symm. PSF	use DCT	use DCT

Easy cases use FFT, DCT or Kronecker product SVD.

General Kronecker Product Decompositions

If the PSF is separable:

$$P = \mathbf{c}\mathbf{d}^T \qquad \text{(rank-1 matrix)}$$

then

$$A = C \otimes D \qquad \text{(Kronecker product)}$$

What to do if PSF is not separable?

General Kronecker Product Decompositions

If the PSF is not separable, we can still compute:

$$P = \sum_{i=1}^{r} \mathbf{c}_i \mathbf{d}_i^T \qquad \text{(sum of rank-1 matrices)}$$

and therefore, get

$$A = \sum_{i=1}^{r} C_i \otimes D_i \qquad \text{(sum of Kron. products)}$$

In fact, we can get "optimal" decompositions.

General Kronecker Product Decompositions

To get "optimal" decomposition, use SVD:

$$P = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

Then

$$\begin{aligned}
\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T &= \text{best rank-1 approximation} \\
\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T &= \text{best rank-2 approximation} \\
&\vdots \\
\sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T &= \text{best rank-r approximation}
\end{aligned}$$

General Kronecker Product Decompositions

Using an optimal decomposition of PSF,

$$P = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \sum_{i=1}^{r} = \mathbf{c}_i \mathbf{d}_i^T$$

where $\quad \mathbf{c}_i = \sqrt{\sigma_i}\mathbf{u}_i, \quad \mathbf{d}_i = \sqrt{\sigma_i}\mathbf{d}_i,$

we get an optimal decomposition of A:

$$A = \sum_{i=1}^{r} C_i \otimes D_i$$

In particular,

$$A \approx C_1 \otimes D_1 = \text{ best Kronecker approximation}$$

General Kronecker Product Decompositions

Summary of Kronecker Product Approximation:

- Given $P =$ PSF, in Matlab we compute:

```
>> [U, S, V] = svd(P);
>> c = sqrt(S(1,1)) * U(:,1);
>> d = sqrt(S(1,1)) * V(:,1);
```

- Depending on the imposed boundary condition,

$$\begin{aligned} C &= \text{Toeplitz, circulant, or Toeplitz+Hankel} \\ D &= \text{Toeplitz, circulant, or Toeplitz+Hankel} \end{aligned}$$

- Computational cost for $n \times n$ images (i.e., $n^2 \times n^2$ A) is $O(n^3)$.

Defining A from PSF

Matrix Summary

BC	PSF Not Separable	PSF Separable
zero	BTTB	use SVD
periodic	use FFT	use FFT
reflexive	BTTB+BTHB +BHTB+BHHB	use SVD
reflexive with symm. PSF	use DCT	use DCT

Easy cases use FFT, DCT or Kronecker product SVD.

Defining A from PSF

Matrix Summary

BC	PSF Not Separable	PSF Separable
zero	use SVD approximation?	use SVD
periodic	use FFT	use FFT
reflexive	use SVD approximation?	use SVD
reflexive with symm. PSF	use DCT	use DCT

Easy cases use FFT, DCT or Kronecker product SVD.

General Kronecker Product Decompositions

SVD Approximations:

- By taking more terms in the PSF decomposition:

  $$P = \mathbf{c}_1 \mathbf{d}_1^T + \mathbf{c}_2 \mathbf{d}_2^T + \cdots + \mathbf{c}_k \mathbf{d}_k^T$$

  we get a better approximation of the PSF.

- This means we can construct a better approximation for A:

  $$A = C_1 \otimes D_1 + C_2 \otimes D_2 + \cdots + C_k \otimes D_k$$

- How to use this? That is, we need an SVD of A, so how do we compute an SVD of this sum?

General Kronecker Product Decompositions

SVD Approximations:

- Construct the approximation:

$$A = C_1 \otimes D_1 + C_2 \otimes D_2 + \cdots + C_k \otimes D_k$$

- Compute an SVD of the dominant term:

$$C_1 \otimes D_1 = (U_c \otimes U_d)(\Sigma_c \otimes \Sigma_d)(V_c \otimes V_d)^T$$

- Determine which of the following gives best approximation of a diagonal matrix:

$$\begin{aligned}
\mathcal{F} A \mathcal{F}^* &= \mathcal{F} \left(\sum C_i \otimes D_i \right) \mathcal{F}^* \\
\mathcal{C} A \mathcal{C}^T &= \mathcal{C} \left(\sum C_i \otimes D_i \right) \mathcal{C}^T \\
(U_c \otimes U_d)^T A (V_c \otimes V_d) &= (U_c \otimes U_d)^T \left(\sum C_i \otimes D_i \right) (V_c \otimes V_d)
\end{aligned}$$

General Kronecker Product Decompositions

SVD Approximations:

Use $A \approx U\Sigma V^T$, (or $A \approx U\Sigma V^*$) where

- If $\mathcal{F}\left(\sum C_i \otimes D_i\right)\mathcal{F}^*$ is best,

$$U = V = \mathcal{F}^*, \quad \Sigma = \text{diag}\left(\mathcal{F}\left(\sum C_i \otimes D_i\right)\mathcal{F}^*\right)$$

- If $\mathcal{C}\left(\sum C_i \otimes D_i\right)\mathcal{C}^T$ is best,

$$U = V = \mathcal{C}^T, \quad \Sigma = \text{diag}\left(\mathcal{C}\left(\sum C_i \otimes D_i\right)\mathcal{C}^T\right)$$

- If $(U_c \otimes U_d)^T\left(\sum C_i \otimes D_i\right)(V_c \otimes V_d)$ is best,

$$U = U_c \otimes U_d, \quad V = V_c \otimes V_d,$$

$$\Sigma = \text{diag}\left((U_c \otimes U_d)^T\left(\sum C_i \otimes D_i\right)(V_c \otimes V_d)\right)$$

Defining A from PSF

Matrix Summary

BC	PSF Not Separable	PSF Separable
zero	try SVD approximation	use SVD
periodic	use FFT	use FFT
reflexive	try SVD approximation	use SVD
reflexive with symm. PSF	use DCT	use DCT

Easy cases use FFT, DCT or Kronecker product SVD.

Preview for Friday

- Kronecker product approximations for spatially variant blurs.

- Accelerating iterative methods.