Fast Algorithms
for the
Removal of Non-Uniform Motion Blurs

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Motivating Example

Motion in MRI cardiac image

Restored MRI cardiac image
• Uniform motion blurs, matrix models

• Non-uniform motion blurs, matrix models

• Algorithms

• Difficulties
Uniform Motion Blur

True Image

Horizontal Blur

Vertical Blur

Angled Blur
The term watershed refers to a ridge that divides areas drained by different river systems.
Linear model: \[ b = Ax + n \]

The matrix \( A \) is structured:

- **Horizontal blur** \[ A = T \otimes I \]
- **Vertical blur** \[ A = I \otimes T \]

\[ T = \begin{bmatrix} \frac{1}{d} & 1 & \cdots & 1 \\ \cdots & \frac{1}{d} & \cdots & \cdots \\ 1 & \cdots & \frac{1}{d} \\ \end{bmatrix} \]

**Angled blur** \[ A = \text{block Toeplitz, Toeplitz blocks} \]
Non-Uniform Motion Blur

True Image

Non-Uniform Blur
We still have linear model

\[ b = Ax + n \]

However, \( A \) has no obvious structure.
Non-Uniform Motion Blur

To explicitly construct $A$:

- Estimate direction and magnitude of motion at each pixel
- Construct corresponding column of $A$
- Use sparse data structure for $A$

Problem: It may be difficult to estimate motion at every pixel
To approximate $A$:

- Partition image into regions
- Assume motion is uniform in each region
- Estimate direction and magnitude of "uniform" motion in each region
- Use interpolation:

$$A \approx \sum I_k A_k$$

where $A_k$ is defined by uniform motion in $k$th region and $I_k$ is diagonal, with $\sum I_k = I$
Region Partitioning

One region
Region Partitioning

64 regions
Given $A$ (or its approximation), need to solve

$$b = Ax + n$$

where

- $A$ is ill-conditioned
  $\Rightarrow$ need regularization

- $A$ is large
  $\Rightarrow$ usually need iterative method
Deblurring Algorithms

Possible regularization methods

- **Truncated singular value decomposition (TSVD)**

  \[ A = UV^T \Rightarrow x_{tsvd} = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i \]

  (can use efficiently for horizontal or vertical blurs)

- **Tikhonov:**

  \[ \min \left\{ \| b - Ax \|^2 + \mu^2 \| Lx \|^2 \right\} \]

- **Iterative:** Terminate iterations early
  (e.g., conjugate gradients)
Numerical Examples

First consider text data:

- Use CGLS, iterative regularization.
- For \( A \),
  - Approximate motion on regions
  - Use motion information at every pixel
Numerical Examples

![Graph showing iteration vs. relative error.](image)

The term watershed refers to a ridge that...
Numerical Examples

1 region, 0 iterations

The term watershed refers to a ridge that...
Numerical Examples

64 region, 3 iterations
Numerical Examples

1024 region, 15 iterations

The term watershed refers to a ridge that ...

... divides areas drained by different river systems.
Numerical Examples

The term watershed refers to a ridge that...
Computational Issues

- Choosing a regularization parameter

- Fast matrix-vector multiplication
  should scale well with number of regions

- Preconditioning
  especially when using a lot of regions, or all pixel information

- How to get motion information?
Using the model: \[ A = \sum_{k=1}^{p} I_k A_k \]

- Each \( A_k \) can be decomposed as:

\[ A_k = \sum_{j=1}^{r} C_j^{(k)} \otimes D_j^{(k)} \]

- Therefore, our model for \( A \) is:

\[ A = \sum_{k=1}^{p} I_k \left( \sum_{j=1}^{r} C_j^{(k)} \otimes D_j^{(k)} \right) \]
• It is possible to efficiently implement iterative methods for non-uniform motion blurs.

• More information provides substantially better restorations.

• However, more information results in slower convergence of iterative methods.

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