1. One important source of error that can arise from using floating point arithmetic is referred to as catastrophic cancellation. This occurs when two nearly equal numbers are subtracted. For example, suppose we want to compute \( x - y \), but because the computer cannot store numbers with infinite precision, and because possibly previous computations (which introduce roundoff errors) were needed to first compute \( x \) and \( y \), we only have approximations \( \hat{x} \approx x \) and \( \hat{y} \approx y \). But we will suppose that \( \hat{x} \) and \( \hat{y} \) are good approximations of \( x \) and \( y \). In particular, we assume

\[
\hat{x} \approx x, \quad \text{with (small) error } \frac{|\hat{x} - x|}{|x|},
\]

\[
\hat{y} \approx y, \quad \text{with (small) error } \frac{|\hat{y} - y|}{|y|}.
\]

If we use \((\hat{x} - \hat{y})\) as an approximation of the true value \((x - y)\), then the relative error is:

\[
\left| \frac{\hat{x} - \hat{y} - (x - y)}{(x - y)} \right|
\]

(a) Show that

\[
\left| \frac{\hat{x} - \hat{y} - (x - y)}{(x - y)} \right| \leq \frac{|\hat{x} - x|}{|x|} \frac{|x|}{|x - y|} + \frac{|\hat{y} - y|}{|y|} \frac{|y|}{|x - y|}
\]

(Hint: You will need to use the triangle inequality.)

(b) From the above relation, explain why the error in computing \( x - y \) can be large if \( x \approx y \), even if \( \hat{x} \) and \( \hat{y} \) are good approximations of, respectively, \( x \) and \( y \).

2. Suppose we want to evaluate the function

\[
f(x) = \frac{1 - \cos x}{x^2}.
\]

(a) Evaluate the following limit:

\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2}
\]

Explain how you found this limit. For example, did you use any particular rules from Calculus?

(b) In MATLAB, create a script m-file called CatCanExample.m, with the following. In your own words, on a clean sheet of paper, explain what each of the commands do in this script.
This script investigates what can happen when catastrophic cancellation occurs when computing values of a simple function.

```matlab
f = @(x) (1 - cos(x))./(x.^2);
n = 0:11;
x = 1.2*10.^(-n);
y = f(x);
rel_error = abs(y - 0.5)/0.5;

fprintf(1, ' x f(x) |f(x) - 0.5|/0.5 
');
fprintf(1, ' ---------------- ---------------- ---------------- 
');
for k = 1:length(x)
    fprintf(1, '%16.8e %16.8e %16.8e 
', x(k), y(k), rel_error(k));
end

figure(1), clf
axes('FontSize', 18)
plot(x, y, 'b-o', 'LineWidth', 2)

figure(2), clf
axes('FontSize', 18)
semilogx(x, y, 'b-o', 'LineWidth', 2)
```

(c) In the above code, explain why it is important to use the operations: ./ and .^.

(d) For this problem, why is the plotting command `semilogx` better than `plot`?

(e) If the code runs successfully, it should produce a table of results. Explain what the results in this table, and in Figure 2 tell you.

3. Using an appropriate identity from trigonometry, show that

\[ f(x) = \frac{1 - \cos(x)}{x^2} = \frac{2\sin^2(x/2)}{x^2}. \]

Be sure to state which trig identity you used.

4. Copy the code from problem 2 into a new script m-file, `NoCatCanExample.m`, and modify it so that it uses

\[ 2\frac{\sin^2(x/2)}{x^2} \]

to evaluate \( f(x) \). Run the code, and comment on your results. In particular, explain why the results are different from what you obtained in problem 2.
What you turn in should be neatly written and organized. You should include your table of results, and printed copies of your figures so that you can refer to them in your solutions. You should also include printed versions of your code. Part of your grade will be based on overall presentation – if you turn in sloppy work, points will be deducted.