Math 351
Spring, 2008

Practice Problems for Section 2.3

1. Consider the homogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\,, 0 < x < \pi\,, \quad t > 0$$

Use the results from the separation of variables method discussed in section 2.3 to find the solution of this PDE for the following initial and boundary conditions.

(a) $u(x,0) = A$ (a constant), $\quad u(0,t) = u(\pi,t) = 0$

(b) $u(x,0) = \sin^3(x)\,, \quad u(0,t) = u(\pi,t) = 0.$ $\qquad$ (Hint: $\sin^3(x) = (3\sin(x) - \sin(3x))/4$)

(c) $u(x,0) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}\,, \quad u(0,t) = u(\pi,t) = 0$

2. From the Haberman text, do Exercise 2.3.1, page 55.

To get you started, consider part (a). The unknown function here is $u(r,t)$. In separation of variables, we want to find solutions that separate the variables; that is, solutions of the form:

$$u(r,t) = \phi(r)G(t)$$

Using this, we find that:

$$\frac{\partial u}{\partial t} = \phi(r)\frac{dG}{dt}\,, \qquad \frac{\partial u}{\partial r} = \frac{d\phi}{dr}G(t)\,, \qquad \frac{\partial^2 u}{\partial r^2} = \frac{d^2\phi}{dr^2}G(t)$$

Substituting these into the given PDE:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{k}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right)$$

we obtain

$$\phi(r)\frac{dG}{dt} = \frac{k}{r}\left(\frac{d\phi}{dr}G(t) + r\frac{d^2\phi}{dr^2}G(t)\right)$$

Using algebra, manipulate this equation so that terms with t and $G(t)$ are on the left side, and terms with r and $\phi(r)$ are on the right side:

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{r\phi}\left(\frac{d\phi}{dr} + r\frac{d^2\phi}{dr^2}\right) = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$$

3. From the Haberman text, do Exercise 2.3.3, page 55.

4. From the Haberman text, do Exercise 2.3.5, page 56.