**Math 351**
**Spring, 2008**

Practice Problems for Section 2.4

1. Consider the the PDE:

$$\begin{aligned} &u_t = k u_{xx}\,, \qquad 0 \le x \le L \quad t \ge 0 \\ &u_x(0,t) = 0 \\ &u_x(L,t) = 0 \\ &u(x,0) = f(x) \end{aligned}$$

   (a) Use separation of variables to reduce the PDE to the two ODEs:

$$\frac{dG}{dt} = -\lambda k G \qquad \text{and} \qquad \begin{aligned} &\frac{d^2\phi}{dx^2} = -\lambda\phi \\ &\frac{d\phi}{dx}(0) = 0 \\ &\frac{d\phi}{dx}(L) = 0 \end{aligned}$$

   (b) Solve the ODEs in part (a). Note that in the boundary value problem, you should find that "case $\lambda > 0$" and "case $\lambda = 0$" lead to nontrivial solutions, but that "case $\lambda < 0$" leads only to the trivial solution. Thus only cases $\lambda > 0$ and $\lambda = 0$ are important.

   (c) Use part (b) to show that the general solution of the PDE is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

   (d) Using the orthogonality condition:

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \ne m \\ \frac{L}{2} & n = m \ne 0 \\ L & n = m = 0 \end{cases}$$

   show that

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots \end{aligned}$$

2. The previous problem shows that the PDE

$$
\begin{aligned}
&u_t = k u_{xx}\,, \qquad 0 \le x \le L \quad t \ge 0 \\
&u_x(0,t) = 0 \\
&u_x(L,t) = 0 \\
&u(x,0) = f(x)
\end{aligned}
$$

has the solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

where

$$
\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L f(x)dx \\
A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots
\end{aligned}
$$

Use this to solve the PDEs in exercise 2.4.1, page 69 of the Haberman book.