

Math 351
Spring, 2008

Practice Problems for Section 9.2
Corrected version – corrections is **red**.

Some comments:

- Recall Green's formula for

$$L(y) = \frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy, \quad 0 < x < L$$

is

$$\int_0^L (uL(v) - vL(u)) dx = \left[p \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right]_0^L$$

- Thus, Green's formula for

$$L(y) = \frac{d^2y}{dx^2}, \quad 0 < x < L$$

is

$$\int_0^L (uL(v) - vL(u)) dx = \left[u \frac{dv}{dx} - v \frac{du}{dx} \right]_0^L$$

or, this can be written as

$$\int_0^L \left(u \frac{d^2v}{dx^2} - v \frac{d^2u}{dx^2} \right) dx = \left[u \frac{dv}{dx} - v \frac{du}{dx} \right]_0^L$$

- Similarly, Green's formula for

$$L(y) = \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L$$

is

$$\int_0^L \left(u \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 u}{\partial x^2} \right) dx = \left[u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right]_0^L \tag{1}$$

Problems: Before doing problems in the text, we first need to do some preliminary work:

1. Consider the PDE: $u_t = ku_{xx} + Q(x, t)$

Suppose $u(x, t)$ can be written as an eigenfunction expansion:

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \quad (2)$$

where

$$\int_0^L \phi_n(x)\phi_m(x)dx = 0 \quad n \neq m$$

(a) Show that

$$a'_n(t) = \frac{\int_0^L (ku_{xx} + Q(x, t))\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}$$

Hints:

- Compute u_t using equation (2), and substitute into PDE. (Only substitute u_t , and leave u_{xx} written as u_{xx}).
 - Multiply both sides of the resulting equation by $\phi_m(x)$ and use orthogonality to simplify.
 - Solve for $a'_m(t)$.
- (b) Suppose $Q(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$, and use orthogonality to find an expression for $q_n(t)$ in terms of $Q(x, t)$ and $\phi_n(x)$.
- (c) Use part (b) to rewrite $a'_n(t)$ as

$$a'_n(t) = q_n(t) + \frac{\int_0^L ku_{xx}\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}$$

2. Now do problems 9.2.1 (a) and (c) in the Haberman text.

Hints for part (a):

- As in the previous problem, use the eigenfunction expansion method to make a guess for $u(x, t)$. Here you should know the eigenfunctions.
- Compute only u_t in the eigenfunction expansion, and substitute into PDE.
- From the previous problem, find expressions for $a'_n(t)$ and $q_n(t)$.
- Use Green's formula (1) with $u = u(x, t)$ and $v = \sin\left(\frac{n\pi x}{L}\right)$ to show that

$$\int_0^L u_{xx} \sin\left(\frac{n\pi x}{L}\right) dx = -\left(\frac{n\pi}{L}\right)^2 \int_0^L u(x, t) \sin\left(\frac{n\pi x}{L}\right) dx$$

- You should now be able to get:

$$a'_n(t) = q_n - k \left(\frac{n\pi}{L}\right)^2 \frac{2}{L} \int_0^L u(x, t) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Explain why this then leads to the ODE:

$$a'_n(t) + k \left(\frac{n\pi}{L}\right)^2 a_n(t) = q_n \tag{3}$$

- The rest of this problem is then done using the same approach as was done in the notes for section 9.2. That is, use the integrating factor method to solve the ODE in equation (3), and follow the notes from section 9.2 to get the Green's function.

3. Do problem 9.2.3 in the Haberman text.

- From the first test, we know the eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

- Guess a solution: $u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$.

- Show that the initial conditions imply:

$$a_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad a'_n(0) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Write $Q(x, t)$ as a Fourier sine series, and find an expression for its coefficients, $q_n(t)$.
- Substitute the guess for $u(x, t)$, and the Fourier sine series for $Q(x, t)$, into the PDE, and show that it leads to the ODE:

$$a''_n(t) + \left(\frac{cn\pi}{L}\right)^2 a_n(t) = q_n(t).$$

- The general solution for this ODE is:

$$a_n(t) = c_1 \cos\left(\frac{n\pi ct}{L}\right) + c_2 \sin\left(\frac{n\pi ct}{L}\right) + \frac{L}{n\pi c} \int_0^t q_n(t_0) \sin\left(\frac{n\pi c(t-t_0)}{L}\right) dt_0$$

where c_1 and c_2 are constants. Use the initial conditions to show that

$$c_1 = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad c_2 = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Now substitute $a_n(t)$, and $q_n(t_0)$, into the guess for $u(x, t)$. Keep in mind that in the integral you will need to change variable notation (such as changing x to x_0). You should be able to show that

$$\begin{aligned} u(x, t) = & \int_0^L f(x_0) \frac{\partial G}{\partial t}(x, x_0; t, 0) dx_0 + \int_0^L g(x_0) G(x, x_0; t, 0) dx_0 \\ & + \int_0^L \int_0^t Q(x_0, t_0) G(x, x_0; t, t_0) dt_0 dx_0 \end{aligned}$$

where

$$G(x, x_0; t, t_0) = \sum_{n=1}^{\infty} \frac{2}{n\pi c} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c(t-t_0)}{L}\right)$$