

Math 351
Spring, 2008

Partial solutions for: Practice Problems for Section 2.3

1. We found in section 2.3, using the method of separation of variables, that the homogeneous heat equation with homogeneous Dirichlet boundary conditions has the solution

$$\begin{aligned}u(x, t) &= \sum_{i=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t} \\ &= \sum_{i=1}^{\infty} B_n \sin(nx) e^{-kn^2 t} \quad (\text{here } L = \pi)\end{aligned}$$

where

$$\begin{aligned}B_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx \quad (\text{here } L = \pi)\end{aligned}$$

and $f(x)$ is specified by the initial conditions, $u(x, 0) = f(x)$.

- (a) If $f(x) = A$, use the integral formula to find:

$$B_n = \frac{2}{\pi} \int_0^{\pi} A \sin(nx) dx = \dots = \frac{2A}{\pi n} (1 - (-1)^n)$$

So, the solution of the PDE is:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2A}{\pi n} (1 - (-1)^n) \sin(nx) e^{-kn^2 t} = \sum_{n=1}^{\infty} \frac{4A}{\pi(2n-1)} \sin((2n-1)x) e^{-k(2n-1)^2 t}$$

- (b) If $u(x, 0) = \sin^3(x) = \frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x)$ then we can see by inspection that

$$B_1 = \frac{3}{4}, \quad B_3 = -\frac{1}{4}, \quad \text{and all other } B_n = 0.$$

Alternatively, we could use the integration formula to find B_n . The orthogonality property of sines (see page 50 of the Haberman text) shows that all B_n are zero except B_1 and B_3 . In any case, the solution of the PDE is:

$$u(x, t) = \frac{3}{4} \sin(x) e^{-kt} - \frac{1}{4} \sin(3x) e^{-9kt}$$

(c) If $u(x, 0) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi \end{cases}$, $u(0, t) = u(\pi, t) = 0$ then

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \left(\int_0^{\pi/2} f(x) \sin(nx) dx + \int_{\pi/2}^\pi f(x) \sin(nx) dx \right) \\ &= \frac{2}{\pi} \left(\int_0^{\pi/2} x \sin(nx) dx + \int_{\pi/2}^\pi (\pi - x) \sin(nx) dx \right) \\ &\quad \vdots \\ &= \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Thus, the solution of the PDE is

$$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin(nx) e^{-kn^2 t} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin((2n-1)x) e^{-k(2n-1)^2 t}$$

2. See solutions given in Haberman text.
3. This problem is essentially the same as the first problem in this set of practice problems.

(a) By inspection, can see that $B_9 = 6$ and all other B_n are zero. Thus

$$u(x, t) = 6 \sin\left(\frac{9\pi x}{L}\right) e^{-k(9\pi/L)^2 t}$$

(b) By inspection, can see that $B_1 = 3$, $B_3 = -1$, and all other $B_n = 0$. Thus

$$u(x, t) = 3 \sin\left(\frac{\pi x}{L}\right) e^{-k(\pi/L)^2 t} - \sin\left(\frac{3\pi x}{L}\right) e^{-k(3\pi/L)^2 t}$$

(c) Here we need to use the integral formula to find B_n , but the integral doesn't come to a neat conclusion, so we won't compute it. Thus, we'll just say that the solution of the PDE is given by

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

where B_n is

$$B_n = \frac{4}{L} \int_0^L \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

(d) This is similar to part (c) of the first problem. So break up the integral, and calculate to find:

$$B_n = \frac{2}{n\pi} \left(1 + \cos\left(\frac{n\pi}{2}\right) - 2 \cos(n\pi) \right)$$

and the solution of the PDE is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 + \cos\left(\frac{n\pi}{2}\right) - 2 \cos(n\pi) \right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

4. See solution on quiz 2.