Math 351
Spring, 2008

Practice Problems for Section 2.4

1. (a) Set $u(x,t) = \phi(x)G(t)$, substitute into PDE, and manipulate to get the specified ODEs.
   (b) When you solve for $\phi(x)$ you should find the eigenvalues and corresponding eigenfunctions:

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad \phi(x) = \cos(\sqrt{\lambda}x) = \cos\left(\frac{n\pi x}{L}\right)$$

$$\lambda = 0, \qquad \phi(x) = \text{constant}$$

   When you solve for $G(t)$ you should get the solutions:

$$G(t) = Ce^{-\lambda kt}.$$

   Thus,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad \Rightarrow \qquad G(t) = e^{-k(n\pi/L)^2 t}$$

$$\lambda = 0, \qquad \Rightarrow \qquad G(t) = \text{constant}$$

   (c) The solution is obtained by forming linear combinations of the eigenfunctions.

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

   (d) Since

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

   consider the integral:

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

   Now look at two cases. First, if $m = 0$, then $\cos(\left(\frac{m\pi x}{L}\right) = 1$, so the equation can be simplified a bit. Then integrate to find

$$\int_0^L f(x)dx = A_0 L \quad \Rightarrow \quad A_0 = \frac{1}{L}\int_0^L f(x)dx$$

   If $m \neq 0$, use orthogonality of cosines to show that

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\right) \cos\left(\frac{m\pi x}{L}\right) dx = \cdots = A_m \frac{L}{2}$$

   Thus,

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx$$

2. (a) Use the integral formulas to find:

$$A_0 = \frac{1}{2}\,, \qquad A_n = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus,

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}$$

(b) By inspection, we see that

$$A_0 = 6\,, \qquad A_3 = 4\,, \qquad \text{and remaining } A_n = 0$$

Thus,

$$u(x,t) = 6 + 4\cos\left(\frac{3\pi x}{L}\right)e^{-k(3\pi/L)^2 t}$$

(c) Use the integral formulas to find

$$A_0 = -\frac{4}{\pi}\,, \qquad A_n = -\frac{4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

The integral for A_n does not simplify easily, so leave it in this form. The solution is then:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)e^{-k(n\pi/L)^2 t}$$

(d) By inspection we see that

$$A_0 = 0\,, \qquad A_8 = -3\,, \qquad \text{and all other } A_n = 0.$$

Thus, the solution is

$$u(x,t) = -3\cos\left(\frac{8\pi x}{L}\right)e^{-k(8\pi/L)^2 t}$$