(a) Find (if possible) nontrivial solutions of
\[
\frac{d^2 \phi}{dx^2} + \phi = 0, \quad \phi(0) = 0, \quad \phi(\pi) = 0.
\]
- Try \( \phi(x) = e^{ix} \), and find that \( r = \pm i \) (imaginary roots).
- This means the general solution of \( \frac{d^2 \phi}{dx^2} + \phi = 0 \) is
  \[
  \phi(x) = c_1 \sin(x) + c_2 \cos(x)
  \]
- Using the BCs, show that \( c_2 = 0 \), and \( c_1 \) is arbitrary.
  \[
  \therefore 2, 0 \text{ is an eigenvalue with eigenfunction } \phi(x) = \sin(x)
  \]
- The problem \( \frac{d^2 u}{dx^2} + u = f(x) \) has infinitely many solutions if
  \[
  \int_0^\pi f(x) \phi_1(x) \, dx = 0
  \]
  In this case, \( f(x) = \sin(x) \), \( \phi_1(x) = \sin(x) \), so
  \[
  \int_0^\pi \sin^2(x) \, dx = \frac{\pi}{2} \neq 0
  \]
  So there are no solutions to \( \frac{d^2 u}{dx^2} + u = \sin(x) \)
  \( u(0) = 0; u(\pi) = 0 \).

(b) Find (if possible) nontrivial solutions of
\[
\frac{d^2 \phi}{dx^2} + \phi = 0, \quad \phi(0) = \phi'(\pi) = 0
\]
Similar to part (a), we find that \( \lambda = 0 \) is an eigenvalue with eigenfunction \( \phi(x) = \cos(x) \).
Now
\[
\int_0^\pi f(x) \phi_1(x) \, dx = \int_0^\pi \sin(x) \cos(x) \, dx = \ldots = 0
\]
\( \therefore \) there are infinitely many solutions to
\[
\frac{d^2 u}{dx^2} + u = \sin(x)
\]
\( u(0) = u'(\pi) = 0 \).
(c) Find (if possible) nontrivial solutions of
\[ \frac{d^2\phi}{dx^2} - \phi = 0, \quad \phi(0) = 0, \quad \phi(\pi) = 0. \]

- Try \( \phi = e^{rx} \), and find that \( r = \pm 1 \) (real roots).
- This means that the general solution of \( \frac{d^2\phi}{dx^2} - \phi = 0 \) is
  \[ \phi(x) = c_1 e^x + c_2 e^{-x}. \]
- Using the BCs, show that you must get \( c_1 = 0 \) and \( c_2 = 0 \).
- Thus, there are no nontrivial solutions of \( \frac{d^2\phi}{dx^2} - \phi = 0 \).
  \[ \phi(0) = 0, \quad \phi(\pi) = 0. \]

(d) Find (if possible) nontrivial solutions of
\[ \frac{d^2\phi}{dx^2} + 2\phi = 0, \quad \phi(0) = 0, \quad \phi(\pi) = 0 \]

- Try \( \phi = e^{rx} \), and find that \( r = \pm \sqrt{2} i \) (imaginary roots).
- This means that the general solution of \( \frac{d^2\phi}{dx^2} + 2\phi = 0 \) is
  \[ \phi(x) = c_1 \sin(\sqrt{2}x) + c_2 \cos(\sqrt{2}x). \]
- Using the BCs, show that you must get \( c_1 = c_2 = 0 \).
- Thus, there are no nontrivial solutions of \( \frac{d^2\phi}{dx^2} + 2\phi = 0 \).
  \[ \phi(0) = 0, \quad \phi(\pi) = 0. \]
- Thus, there are no new eigenvalues, and so there is one, unique solution of
  \[ \frac{d^2u}{dx^2} + 2u = \sin(x), \quad u(0) = 0, \quad u(\pi) = 0. \]
\[ \frac{d^2u}{dx^2} + u = \beta + x, \quad u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi) \]

- Try to find a nontrivial solution of:
  \[ \frac{d^2u}{dx^2} + u = 0 \]

  The general solution of this ODE is (see solutions for problem 41):
  \[ \phi(x) = c_1 \sin(x) + c_2 \cos(x) \]

- The boundary conditions do not eliminate any of the constants \( c_1, c_2 \).
  \[ \therefore \text{The general solution holds for all } c_1, c_2. \]

- Thus, there are many nontrivial solutions.
  In particular, \( \lambda = 0 \) is an eigenvalue with corresponding eigenfunction
  \[ \phi(x) = \sin(x) + \cos(x) \]

- We want to know if there are any \( \beta \) such that:

  \[ \int_{-\pi}^{\pi} (\beta + x) [\sin(x) + \cos(x)] \, dx = 0. \]

  This integral is always \( 2\pi \), for any \( \beta \).

  \[ \therefore \text{There } \text{no } \beta \text{ such that} \]

  \[ \int_{-\pi}^{\pi} (\beta + x) [\sin(x) + \cos(x)] \, dx = 0 \]

  and so there are no \( \beta \) for which this problem has a solution.