Algebraic groups over $K = \overline{K}$ (Lecture III)
Chevalley’s theorem, Jordan decomposition

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Let $G$ act on affine varieties $X$ and $Y$ via $\phi_X$ and $\phi_Y$.

Then $f : X \to Y$ is said to be a $G$-equivariant morphism if it is a variety morphism such that the following diagram commutes:

$$
\begin{array}{c}
G \times X & \xrightarrow{\phi_X} & X \\
\downarrow{(id_X,f)} & & \downarrow{f} \\
G \times Y & \xrightarrow{\phi_Y} & Y
\end{array}
$$
Recall: Dual representations

- Let $H$ be a finite group and let it act on a vector space $V/K$.
- Then $H$ acts on $V^* = \text{Hom}_K(V, K)$ as follows:

$$H \times V^* \to V^*$$

$$(h, f) \mapsto [v \mapsto f(h^{-1}v)]$$

- There is an inverse in the above definition to make it a left action.
The left dual representation of $G$

- $G$ is an affine algebraic group over $K$
- Let it act on the left on an affine variety $X$ via 
  $\phi_X : G \times X \to X, (g, x) \mapsto gx$
- Think of elements $f$ in the coordinate ring $K[X]$ as regular morphisms $f : X \to K$.
- Then $G$ acts on its coordinate $V = K[X]$ as follows
  $$\lambda : G \times K[X] \to K[X] \text{ where } (g, f) \mapsto [x \mapsto f(g^{-1}x)]$$
- The corresponding dual representation is given by
  $$\lambda : G \to \text{GL}(K[X])$$
  $$g \mapsto \lambda_g = [x \mapsto f(g^{-1}x)]$$
- Caution: If $V$ is not finite dimensional, then $\text{GL}(V)$ is not an affine variety. Thus $\text{GL}(K[X])$ is not an affine variety.
Chevalley’s theorem: Jordan decomposition

The right dual representation of $G$

- $G$ is an affine algebraic group over $K$
- Let it act on the right on an affine variety $Y$ via
  $\phi_Y : Y \times G \to Y, (y, g) \mapsto y \star g$
- We can convert it into a left action by defining $gy = y \star g^{-1}$
- Think of elements $f$ in the coordinate ring $K[Y]$ as regular morphisms $f : Y \to K$.
- Then $G$ acts on its coordinate $V = K[Y]$ as follows
  $\rho : G \times K[Y] \to K[Y]$ where $(g, f) \mapsto [y \mapsto f(g^{-1}y) = f(y \star g)]$
- The corresponding dual representation is given by
  $\rho : G \to GL(K[Y])$
  $g \mapsto \rho_g = [x \mapsto f(y \star g)]$
The two dual representations of $G$

- $G$ is an affine algebraic group over $K$.
- It acts on itself via left and right translations!

$$\ell_G : G \times G \to G$$

$$g,x \leadsto gx$$

$$r_G : G \times G \to G$$

$$y,g \leadsto yg$$

$G$ acts on its coordinate $V = K[G]$ by the dual representations

$$\lambda : G \times K[G] \to K[G] \text{ where } (g,f) \leadsto [x \to f(g^{-1}x)]$$

$$\rho : G \times K[G] \to K[G] \text{ where } (g,f) \leadsto [y \to f(yg)]$$
The dual representations are injective

- Let us show it for the $\lambda$ representation

$$\lambda : G \rightarrow \text{GL}(K[G]) \quad \text{where} \quad g \mapsto (f \mapsto [x \mapsto f(g^{-1}x)])$$

- Let $\lambda(g) = \lambda(g')$
- Then $f(g^{-1}x) = f(g'^{-1}x)$ for all $f \in K[G]$ and for all $x \in G$
- This implies $g^{-1}x = g'^{-1}x$ for all $x \in G$
- Hence $g = g'$!
An example when $G = \mathbb{G}_a$

- Let $G = \mathbb{G}_a$
- Then $K[G] = K[t]$
- Let $g \in G(K) = K$
- Let us understand how $g$ acts on $K[t]$ in the left dual representation
- That is, let us write down what $\lambda_g$ is

$$
\lambda_g : K[t] \rightarrow K[t] \\
f \mapsto [x \rightarrow f(xg^{-1})] \\
\quad f \mapsto [x \rightarrow f(x - g)] \\
\quad f \mapsto f(t - g)
$$

- Similarly $\rho_g(f(t)) = f(t + g)$
Approximating $K[t]$ by $G$-subspaces

- Now $K[t]$ is an infinite dimensional $K$ vector space
- Thus $\lambda : G \rightarrow \text{GL}(K[t])$ is an infinite dimensional representation of $G$
- We would like to approximate this representation by finite dimensional representations!
- So we would like to break up $K[t]$ into finite dimensional $G$-stable subspaces.
Approximating $K[t]$ by $G$-subspaces

- Consider $F_i$, the $K$ vector space of polynomials of degree $\leq i$.
- This is a finite dimensional vector space.
- And it is $G$-stable for the $\lambda$ action.
- This is because $gf(t) = f(t - g)$ is a polynomial again of the same degree!
- Thus $K = F_0 \subseteq F_1 \subseteq F_2 \ldots$ and $K[t] = \bigcup_{i=0}^{\infty} F_i$.
- We have approximated the representation $\lambda$ by finite dimensional representations!
- In fact, we have found a nice $G$-stable flag for $K[t]$ under the dual action for the left translation action of $G$. 
Want to generalize this process for an arbitrary $G$ acting on itself by translation.

And get a nice $G$-stable flag for the dual action of $G$ on $K[G]$.

This will actually help us embed $G$ into a general linear group $GL(V)$.

And it will also help in *Jordan decomposition* which breaks up elements of our algebraic group into semisimple and unipotent parts.
Step 1: Finding $G$-stable finite dimensional spaces

- Let $F$ be a finite dimensional $K$-subspace of $K[G]$
- $F = Kf_1 \oplus Kf_2 \oplus \ldots Kf_n$
- Goal: Find a $G$-stable finite dimensional subspace $E$ such that

$$F \subseteq E \subseteq K[G]$$

- Instead find finite dimensional $G$-stable subspace $E_i \supseteq Kf_i$
- Then $E = \sum E_i$ works!
- Thus wlog can assume $F = Kf$, i.e. $F$ is one dimensional
Step I: Finding $G$-stable finite dimensional spaces

- Let $F = Kf \subseteq K[G]$
- Goal: Find a $G$-stable finite dimensional subspace $E$ such that $F \subseteq E \subset K[G]$
- We are going to find a finite dimensional vector subspace $W$ which will contain the orbit of $f$
- This will show that the $G$-orbit of $Kf$ is finite dimensional, which will be our $E$
Defining $W$

- To define $W$, recall the co-multiplication $\mu^* : K[G] \rightarrow K[G] \otimes K[G]$
- Let $\mu^*(f) = \sum_{i=1}^{n} h_i \otimes g_i$
- Set $W = \sum_{i=1}^{n} Kg_i$
- Clearly $W$ is finite dimensional $K$-space
- $W$ needn’t be $G$-stable!
What is the $G$-orbit of $f$

- Let $g \in G$
- $f \in K[G]$ and as usual think of it as a regular function $f : G \to K$ or $f : G \to \mathbb{A}^1$
- What is $gf$?

$$\lambda_g : K[G] \to K[G]$$

$$f \sim [x \mapsto f(g^{-1}x)]$$

- Thus $gf : G \to K$ which sends $x \mapsto f(g^{-1}x)$
What is the $G$-orbit of $f$

- $gf$ in the algebraic groups language:

$$gf : G \xrightarrow{(g^{-1},id_G)} G \times G \xrightarrow{\mu} G \xrightarrow{f} \mathbb{A}^1$$

$$x \xrightarrow{} (g^{-1}, x) \xrightarrow{} g^{-1}x \xrightarrow{} f(g^{-1}x)$$

- $gf$ in the Hopf algebras language:

$$(gf)^* : K[G] \xleftarrow{(g^{-1*},id_G^*)} K[G] \otimes K[G] \xleftarrow{\mu^*} K[G] \xleftarrow{f^*} K[t]$$

$$\sum h_i(g^{-1})g_i \xleftarrow{} \sum h_i \otimes g_i \xleftarrow{} f \xleftarrow{} t$$
$W$ contains the $G$-orbit of $f$

- Thus $gf = \sum h_i(g^{-1})g_i$
- Note that $h_i(g^{-1}) \in K$
- Thus $gf \in \sum_i Kg_i = W$
- Thus $gf \in W$ for all $g \in G$
- So $W$ contains the $G$-orbit of $f$
- Since $W$ is finite dimensional, the $G$ orbit of $f$ and $Kf$ is finite dimensional!
Let $F \subseteq K[G]$ be a finite dimensional $K$-subspace.

Our previous discussion shows that if $\mu^*(F) \subseteq K[G] \otimes F$, then $F$ is $G$-stable under the dual representation.

The converse also holds!

Namely let $F$ be $G$-stable. Then let us show $\mu^*(F) \subseteq K[G] \otimes F$.
Step II: Criteria for testing $G$-stability

- Let $\{f_i\}_{i \in I}$ be a $K$ basis of $F$.
- Extend it to a $K$ basis of $K[G]$.
- So $K[G]$ has a $K$ basis $\{f_i\}_{i \in I} \cup \{f_j\}_{j \in J}$.
- For $f \in F$, let $\mu^*(f) = \sum_{k=1}^{x_n} h_k \otimes f_k$ and if possible, let $x_1 \in J$.
- That is $f_{x_1}$ is a part of the extended basis and not in $F$ and $\mu^*(f) \not\in K[G] \otimes F$.
- As we have seen $gf = \sum_{k=x_1}^{x_n} h_k (g^{-1}) f_k$ for any $g \in G$.
- Since $F$ is $G$-stable, $gf \in F$ for all $g \in G$.
- Find a $g$ such that $h_{x_1}(g^{-1}) = 1$.
- Thus $gf = f_{x_1} + \ldots \not\in F$, a contradiction!
We are now ready to prove ...

Theorem (Chevalley)

If $G$ is an affine algebraic group, then there is a finite dimensional vector space $V$ such that $G$ is isomorphic to a closed subgroup of $GL(V)$. 
Proof of Chevalley’s theorem

- $K[G]$ is a finitely generated algebra
- Let $F$ be the finite dimensional vector space spanned by a (finite) set of $K$-algebra generators of $K[G]$
- By Step I, find a finite dimensional $G$-stable subspace $W$ containing $F$
- Thus $W$ contains a set of $K$-algebra generators of $K[G]$
- Let $W = \oplus_{i=1}^{n} Kw_{i}$, where say the first $\ell$ elements $w_1, w_2, \ldots, w_{\ell}$ form a generating set of $K[G]$ as a $K$ algebra.
Proof of Chevalley’s theorem

- We are going to try and embed $G$ in $GL(W)$
- Since $W$ is $G$-stable, by Step II we know that for $i \leq n$
  \[ \mu^*(w_i) = \sum_j m_{ij} \otimes w_j \]
- Here $m_{ij} \in K[G]$
- Thus we have the $G$ action on $W$
  \[ G \times W \to W \]
  \[ g, w_i \mapsto \sum_j m_{ij}(g^{-1})w_j \]
- Thus we have a map
  \[ \phi : G \to GL(W), \quad g \mapsto (m_{ij}(g^{-1})) \]
\( \phi \) is a variety morphism

- \( \phi : G \to \text{GL}(W), \ g \leadsto (m_{ij}(g^{-1})) \)
- Since \( m_{ij} \in K[G] \), they are regular functions on \( G \).
- And since \( i : G \to G, g \leadsto g^{-1} \) is a variety morphism, we see that \( \phi \) is a variety morphism!
\( \phi \) is injective as a topological map

- \( \phi : G \to GL(W), \ g \mapsto (m_{ij}(g^{-1})) \)
- Let \( \phi(g) = \phi(g') \)
- Now \( gw_i = \sum m_{ij}(g^{-1})w_j \) and \( g'w_i = \sum m_{ij}(g'^{-1})w_j \)
- Thus \( gw_i = g'w_i \) for all \( i \)
- But by definition, \( (gw_i)(1) = w_i(g^{-1}) \) and \( g'w_i = w_i(g'^{-1}) \)
- Thus \( w_i(g^{-1}) = w_i(g'^{-1}) \) for all \( i \)
- But \( \{w_i\} \) generate \( K[G] \) as a \( G \)-algebra
- Thus \( f(g^{-1}) = f(g'^{-1}) \) for all \( f \in K[G] \)
- But functions in \( K[G] \) have to separate points of \( G \)
- This implies \( g^{-1} = g'^{-1} \) and hence \( g = g' \)
\( \phi \) is a group homomorphism

- \( \phi : G \to \text{GL}(W), \ g \mapsto (m_{ij}(g)) \)
- Thus need to show \( (m_{ij}(g^{-1})) \ (m_{ij}(h^{-1})) = (m_{ij}((gh)^{-1})) \) for \( g, h \in G \)
- As before \( w_i(g^{-1}x) = gw_i(x) = \sum_j m_{ij}(g^{-1})w_j(x) \) for all \( x \in G \)
\( \phi \) is a group homomorphism

\[
\begin{align*}
w_i(h^{-1}g^{-1}x) &= w_i((gh)^{-1}x) \\
&= (gh)w_i(x) \\
&= \sum_k m_{ik}((gh)^{-1})w_k(x)
\end{align*}
\]

\[
\begin{align*}
w_i(h^{-1}g^{-1}x) &= hw_i(g^{-1}x) \\
&= \sum_j m_{ij}(h^{-1})w_j(g^{-1}x) \\
&= \sum_j m_{ij}(h)(gw_j(x)) \\
&= \sum_j m_{ij}(h^{-1}) \left( \sum_k m_{jk}(g^{-1})w_k(x) \right)
\end{align*}
\]
\( \phi \) is a group homomorphism

- For all \( x \in G \), we have

\[
\sum_k m_{ik}((gh)^{-1})w_k(x) = \sum_j m_{ij}(h^{-1}) \left( \sum_k m_{jk}(g^{-1})w_k(x) \right)
\]

- Thus we have

\[
\sum_k m_{ik}((gh)^{-1})w_k = \sum_j m_{ij}(h^{-1}) \left( \sum_k m_{jk}(g^{-1})w_k \right)
\]

- Comparing coefficients of \( w_k \)

\[
m_{ik}((gh)^{-1}) = \sum_j m_{ij}(h^{-1})m_{jk}(g^{-1})
\]

- \((m_{ik}((gh)^{-1})) = (m_{ij}(h^{-1}))(m_{jk}(g^{-1}))\)
\( \phi \) is \textit{almost} a group homomorphism

- \( (m_{ik}( (gh)^{-1} )) = (m_{ij}( h^{-1} ))(m_{jk}( g^{-1} )) \)
- \( \phi(gh) = \phi(h)\phi(g) \) (unless I’ve messed up inverses before ..)
- Define \( \tilde{\phi} : G \to \text{GL}(W) \) by sending \( g \leadsto \phi(g^{-1}) \)
- Then

\[
\tilde{\phi}(gh) = \phi(h^{-1}g^{-1}) \\
= \phi(g^{-1})\phi(h^{-1}) \\
= \tilde{\phi}(g)\tilde{\phi}(g)
\]

- Thus \( \tilde{\phi} \) is a group homomorphism. It is also a variety morphism which is injective as a topological map.
- This is because the inverse map \( i : G \to G \) is an isomorphism of varieties
\[
\tilde{\phi} \text{ is a closed embedding of } G \text{ into } \text{GL}(W)
\]

- \(\tilde{\phi} : G \to \text{GL}(W), \ g \mapsto (m_{ij}(g))\) is an algebraic group map
- Thus the image of \(\tilde{\phi}\) is a closed subgroup of \(\text{GL}(W)\)
- But we want to show \(\tilde{\phi}\) is a closed embedding
- That is, we want the corresponding \(K\)-algebra map \(K[\text{GL}(W)] \to K[G]\) to be onto
- This is the map

\[
K[T_{ij}, 1/\det] \to K[G]
\]

\[
T_{ij} \mapsto m_{ij}
\]
\( \tilde{\phi} \) is a closed embedding of \( G \) into \( \text{GL}(W) \)

- Want to show that the following map is onto

\[
K[T_{ij}, 1/\det] \rightarrow K[G]
\]

\( T_{ij} \sim m_{ij} \)

- Since it is a \( K \)-algebra map, enough to show \( \{m_{ij}\} \) generate \( K[G] \) as a \( K \)-algebra
- Recall \( \{w_k\} \) generate \( K[G] \) as a \( K \)-algebra!
- Thus enough to show \( w_k \) s are in the \( K \)-algebra generated by \( \{m_{ij}\} \)
And Chevalley is done!

\[ w_k(g) = g^{-1}w_k(1) \]
\[ = \sum_j m_{kj}(g)w_j(1) \]

\[ \Rightarrow w_k = \sum_j w_j(1)m_{kj} \]
\[ \in \sum_j Km_{kj} \]
Some linear algebra

- Let $M \in \text{End}(V)$ where $V$ is a vector space over $k = \overline{k}$
- Let $f(T) = \prod (T - a_i)^{n_i}$ be the char poly of $M$
- By Chinese remainder theorem, find $P(T) \in k[T]$ which is
  - $0 \text{ mod } T$
  - $a_i \text{ mod } (T - a_i)^{n_i}$
- Set $M_s := P(M)$
- Set $M_n := M - M_s = M - P(M)$
- So $M_n := Q(M)$ where $Q(T) = T - P(T) \in k[T]$
- So $M = M_s + M_n = P(M) + Q(M)$
- By construction $M_s, M_n$ commute with $M$ and each other.
Some linear algebra

• Generalized eigen space decomposition:

\[ V = \bigoplus V_i \text{ where } V_i = \{ v \mid (M - a_i)^{n_i}(v) = 0 \} \]

• Each \( V_i \) is \( M \)-stable and hence \( M_s \) and \( M_n \) stable.

• Each \( V_i \) is the eigen space of \( a_i \) for \( M_s \)

• Thus \( M_s \) is semisimple (i.e, diagonalizable)

• \( M_n \) is nilpotent (Check!)

• This is the additive Jordan decomposition for \( M \) into its semisimple and nilpotent parts (which commute with each other)
Uniqueness of Jordan decomposition for a matrix $M$

- Let $M = A_s + A_n$ where $A_s, A_n$ are s.s and nilpotent elements which commute with each other.
- So $A_s$ commutes with $A_s + A_n = M$ and hence also with $M_s$ and $M_n$.
- And $A_n$ commutes with $M_s$ and $M_n$ also.
- $A_s - M_s = M_n - A_n$.
- FACT: Commuting diagonalizable matrices can be diagonalized simultaneously!
- Sum of commuting nilpotent matrices is also nilpotent (Check!)
- So $A_s - M_s$ is both semisimple and nilpotent and hence 0.
Summary: Jordan decomposition for a matrix $M$

**Theorem**

*Given $M \in \text{End}(V)$*

1. *There exist unique elements $M_s, M_n \in \text{End}(V)$ such that $M_s$ is semisimple and $M_n$, nilpotent and*

   $$M_s M_n = M_n M_s$$
   $$M = M_s + M_n$$

2. *There exist $P(T), Q(T) \in k[T]$ such that $T | P$ and $T | Q$ with $M_s = P(M)$ and $M_n = Q(M)$*
Multiplicative Jordan decomposition for an invertible matrix $M$

- $M \in \text{GL}(V)$
- $M = M_s + M_n$ where $M_s$ is also invertible [no 0 eigen values for $M$ and $M_s$!]
- $M = M_s (1 + M_s^{-1} M_n) = M_s M_u$
- $M_u = 1 + M_s^{-1} M_n$ is unipotent [i.e, $M_u - 1$ is nilpotent]
- This is because $M_s$ and $M_n$ commute!
Summary: Multiplicative Jordan decomposition for an invertible matrix $M$

**Theorem**

Given $M \in \text{GL}(V)$, there exist unique elements $M_s, M_u \in \text{GL}(V)$ such that $M_s$ is semisimple and $M_u$, unipotent and

$$M_s M_u = M_u M_s$$

$$M = M_s M_u$$
Compatibility of Jordan decomposition for commuting diagrams

Let $f : V \to W$, $a : V \to V$ and $b : W \to W$ be such that $f \circ a = b \circ f$. Then $f \circ a_s = b_s \circ f$ and $f \circ a_n = b_n \circ f$. 

\[
\begin{array}{ccc}
V & f & W \\
\downarrow a & & \downarrow b \\
V & f & W \\
\end{array} \quad \quad \begin{array}{ccc}
V & f & W \\
\downarrow a_s & & \downarrow b_s \\
V & a_n & W \\
\end{array} \quad \quad \begin{array}{ccc}
V & f & W \\
\downarrow b_n & & \downarrow b_n \\
V & f & W \\
\end{array}
\]
Compatibility of Jordan decomposition for $\oplus$ and $\otimes$

Jordan decompositions for $V$ and $W$ behave well with respect to $V \otimes W$ and $V \oplus W$
Locally finite version

- $V$ is a $K$-vector space [not necessarily finite dimensional]
- An endomorphism $a \in \text{End}(V)$ is locally finite if $V$ is the union of finite dimensional $a$-stable subspaces
- Locally finite $a \in \text{End}(V)$ is semi-simple if $a|_W$ is semi-simple for any $a$-stable f.d subspace $W$
- Locally finite $a \in \text{End}(V)$ is locally nilpotent if $a|_W$ is nilpotent for any $a$-stable f.d subspace $W$
- Locally finite $a \in \text{End}(V)$ is locally unipotent if $a - 1$ is locally nilpotent
- There are additive and multiplicative Jordan decompositions for any locally finite endomorphism $a \in \text{End}(V)$
Application to the dual representation of $G$

- $G$ is an affine algebraic group over $K$
- $G$ acts on its coordinate $V = K[G]$ by the dual representations

\[ \rho : G \times K[G] \to K[G] \text{ where } (g, f) \mapsto [x \to f(xg)] \]
\[ \lambda : G \times K[G] \to K[G] \text{ where } (g, f) \mapsto [x \to f(g^{-1}x)] \]

- Have shown [in the course of proving Chevalley’s theorem] that $\rho(g), \lambda(g)$ are locally finite in $\text{End}(K[G])$ for each $g \in G$
- So let $\rho(g) = \rho(g)_s \rho(g)_u$ be the multiplicative Jordan decomposition in $\text{End}(K[G])$
Abstract this out and get a Jordan decomposition for $g$ in $G$ (which gives the Jordan decomposition for any linear representation of $G \to \text{GL}(W)$)
Abstract Jordan decomposition (for algebraic groups)

Theorem

Let \( g \in G \). Then there exist unique elements \( g_s, g_u \in G \) such that

1. \( \rho(g_s) = \rho(g)_s \)
2. \( \rho(g_u) = \rho(g)_u \)
3. \( g_u g_s = g_s g_u = g \)