Algebraic groups over $K = \bar{K}$ (Lecture IV)

Jordan decomposition, Diagonalizable groups, Character groups

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Some linear algebra

- Let $M \in \text{End}(V)$ where $V$ is a vector space over $k = \overline{k}$
- Let $f(T) = \prod (T - a_i)^{n_i}$ be the char poly of $M$
- By Chinese remainder theorem, find $P(T) \in k[T]$ which is
  - $0 \mod T$
  - $a_i \mod (T - a_i)^{n_i}$
- Set $M_s := P(M)$
- Set $M_n := M - M_s = M - P(M)$
- So $M_n := Q(M)$ where $Q(T) = T - P(T) \in k[T]$
- So $M = M_s + M_n = P(M) + Q(M)$
- By construction $M_s, M_n$ commute with $M$ and each other.
Some linear algebra

- **Generalized eigen space decomposition:**

  \[ V = \bigoplus V_i \text{ where } V_i = \{ v \mid (M - a_i)^{n_i}(v) = 0 \} \]

- Each \( V_i \) is \( M \)-stable and hence \( M_s \) and \( M_n \) stable.

- Each \( V_i \) is the eigen space of \( a_i \) for \( M_s \)

- Thus \( M_s \) is semisimple (i.e., diagonalizable)

- \( M_n \) is nilpotent (Check!)

- This is the *additive* Jordan decomposition for \( M \) into its semisimple and nilpotent parts (which commute with each other)
Uniqueness of Jordan decomposition for a matrix $M$

- Let $M = A_s + A_n$ where $A_s, A_n$ are s.s and nilpotent elements which commute with each other.
- So $A_s$ commutes with $A_s + A_n = M$ and hence also with $M_s$ and $M_n$.
- And $A_n$ commutes with $M_s$ and $M_n$ also.
- $A_s - M_s = M_n - A_n$.
- FACT: Commuting diagonalizable matrices can be diagonalized simultaneously!
- Sum of commuting nilpotent matrices is also nilpotent (Check!)
- So $A_s - M_s$ is both semisimple and nilpotent and hence 0.
Summary: Jordan decomposition for a matrix $M$

Theorem

Given $M \in \text{End}(V)$

1. There exist unique elements $M_s, M_n \in \text{End}(V)$ such that $M_s$ is semisimple and $M_n$, nilpotent and

   $$M_s M_n = M_n M_s$$

   $$M = M_s + M_n$$

2. There exist $P(T), Q(T) \in k[T]$ such that $T | P$ and $T | Q$ with $M_s = P(M)$ and $M_n = Q(M)$
Jordan decomposition
Diagonalizable groups
Character groups

Multiplicative Jordan decomposition for an invertible matrix \( M \)

- \( M \in \text{GL}(V) \)
- \( M = M_s + M_n \) where \( M_s \) is also invertible [no 0 eigen values for \( M \) and \( M_s \)]
- \( M = M_s (1 + M_s^{-1}M_n) = M_s M_u \)
- \( M_u = 1 + M_s^{-1}M_n \) is unipotent [i.e, \( M_u - 1 \) is nilpotent]
- This is because \( M_s \) and \( M_n \) commute!
Summary: Multiplicative Jordan decomposition for an invertible matrix $M$

**Theorem**

Given $M \in \text{GL}(V)$, there exist unique elements $M_s, M_u \in \text{GL}(V)$ such that $M_s$ is semisimple and $M_u$, unipotent and

\[ M_s M_u = M_u M_s \]

\[ M = M_s M_u \]
Compatibility of Jordan decomposition for commuting diagrams

Let \( f : V \rightarrow W \), \( a : V \rightarrow V \) and \( b : W \rightarrow W \) be such that \( f \circ a = b \circ f \). Then \( f \circ a_s = b_s \circ f \) and \( f \circ a_n = b_n \circ f \).
Compatibility of Jordan decomposition for $\oplus$ and $\otimes$

Jordan decompositions for $V$ and $W$ behave well with respect to $V \otimes W$ and $V \oplus W$
Locally finite version

- $V$ is a $K$-vector space [not necessarily finite dimensional]
- An endomorphism $a \in \text{End}(V)$ is locally finite if $V$ is the union of finite dimensional $a$-stable subspaces
- Locally finite $a \in \text{End}(V)$ is semi-simple if $a|_W$ is semi-simple for any $a$-stable f.d subspace $W$
- Locally finite $a \in \text{End}(V)$ is locally nilpotent if $a|_W$ is nilpotent for any $a$-stable f.d subspace $W$
- Locally finite $a \in \text{End}(V)$ is locally unipotent if $a - 1$ is locally nilpotent
- There are additive and multiplicative Jordan decompositions for any locally finite endomorphism $a \in \text{End}(V)$
Application to the dual representation of $G$

- $G$ is an affine algebraic group over $K$
- $G$ acts on its coordinate $V = K[G]$ by the dual representations

$$\rho : G \times K[G] \rightarrow K[G] \text{ where } (g, f) \mapsto [x \rightarrow f(xg)]$$

$$\lambda : G \times K[G] \rightarrow K[G] \text{ where } (g, f) \mapsto [x \rightarrow f(g^{-1}x)]$$

- Have shown [in the course of proving Chevalley’s theorem] that $\rho(g), \lambda(g)$ are locally finite in $\text{End}(K[G])$ for each $g \in G$
- So let $\rho(g) = \rho(g)_s \rho(g)_u$ be the multiplicative Jordan decomposition in $\text{End}(K[G])$
Goal

Abstract this out and get a Jordan decomposition for $g$ in $G$ (which gives the Jordan decomposition for any linear representation of $G \to \text{GL}(W)$)
Abstract Jordan decomposition (for algebraic groups)

Theorem
Let $g \in G$. Then there exist unique elements $g_s, g_u \in G$ such that

1. $\rho(g_s) = \rho(g)_s$
2. $\rho(g_u) = \rho(g)_u$
3. $g_u g_s = g_s g_u = g$
Proof

- Note that $\rho(g) : K[G] \to K[G]$ is a $K$-algebra automorphism, and not just $K$ linear

- Whereas Jordan decomposition $\rho(g) = \rho(g)_s \rho(g)_u$ only gives $\rho(g)_s \in \text{End}(K[G])$ [just $K$ linear map of vector spaces]

- We would like to find an element $g_s \in G$ which under $\rho$ goes to $\rho(g)_s$, in which case $\rho(g)_s$ will be a $K$-algebra map

- So the first step is to prove $\rho(g)_s$ is a $K$-algebra map.
Proof


Since $\rho(g)$ is a $K$-algebra map, this diagram commutes

$$
\begin{array}{ccc}
A \otimes A & \rightarrow & A \\
\downarrow & & \downarrow \\
\rho(g) \otimes \rho(g) & \rightarrow & \rho(g) \\
\downarrow & & \downarrow \\
A \otimes A & \rightarrow & A \\
\end{array}
$$
Proof

Since Jordan decomposition is compatible with commuting diagrams, the diagram below commutes

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{mult} & A \\
\downarrow & & \downarrow \\
\rho(g)_s \otimes \rho(g)_s & \xrightarrow{\rho(g)_s} & \rho(g)_s \\
\end{array}
\]

This shows $\rho(g)_s : A \rightarrow A$ is a $K$ algebra map
Proof

• If there exists $g_s$ in $G$ such that $\rho(g)_s = \rho(g_s)$, then

$$\rho(g)_s(f)(x) = f(xg_s)$$
$$\rho(g)_s(f)(e) = f(g_s)$$

• To recover $g_s$, define map

$$g_s^* : A \to k$$

$$f \sim f(g_s)$$
$$f \sim \rho(g)_s(f)(e)$$

• This is an algebra map because $\rho(g)_s$ is a $K$-algebra map

• Hence $g_s^*$ defines the point $g_s$!
Proof

• Still need to verify that $\rho(g) s(f)(x) = f(xg_s)$
• $\lambda(h)$ and $\rho(g)$ commute for all $g, h \in G$ because

\[
\rho(g)[\lambda(h)(f)](x) = (\lambda(h)(f))(xg) \\
= f(h^{-1}xg)
\]

\[
\lambda(h)[\rho(g)(f)](x) = (\rho(g)(f))(h^{-1}x) \\
= f(h^{-1}xg)
\]

• So $\lambda(h)$ and $\rho(g)_s$ commute for all $g, h \in G$!
Proof

\[
[\rho(g)_s(f)](x) = [\rho(g)_s(f)]((x^{-1})^{-1}e) \\
= \lambda(x^{-1})[\rho(g)_s(f)](e) \\
= \rho(g)_s[\lambda(x^{-1})(f)](e) \\
= g^*_s (\lambda(x^{-1})(f)) \\
= [\lambda(x^{-1})(f)](g_s) \\
= f(xg_s)
\]
Proof

• Set $g_u = g_s^{-1}g$. Then

$$\rho(g) = \rho(g_s g_u)$$
$$= \rho(g_s) \rho(g_u)$$
$$= \rho(g)_s \rho(g_u)$$

• But $\rho(g) = \rho(g)_s \rho(g)_u$ too
• Hence $\rho(g_u) = \rho(g)_u$
• $g_s$ and $g_u$ commute because $\rho(g)_s$ and $\rho(g)_u$ commute and $\rho : G \to GL(K[G])$ is injective!
• Uniqueness of $g_s$ and $g_u$ follows uniqueness of Jordan decomposition of $\rho(g)$ and $\rho$ being injective.
Abstract Jordan decomposition matches with matrix Jordan decomposition

**Theorem**

Let $G = \text{GL}_n$ and let $g \in G$. Then the abstract Jordan decomposition $g = g_sg_u$ is the same as the matrix Jordan decomposition for the matrix $g \in \text{GL}_n$. That is,

- $g_s$ is just the semisimple part of the matrix $g$
- $g_u$ is just the unipotent part of the matrix $g$
Proof

• Let $g = a_s a_u$ be the matrix Jordan decomposition in $\text{GL}(V)$
• Let $f \in \text{Hom}_K(V, K)$ be a linear functional on $V$
• Define $K$ linear map, $\tilde{f} : V \to K[G]$ sending $v \to [g \leadsto f(gv)]$

• $\tilde{f}$ is injective! [If not, $f(gv) = 0 \forall g \in \text{GL}(V)$ which is not happening unless $v = 0$]
Proof

\[ \rho(g)[\tilde{f}(v)](h) = [\tilde{f}(v)](hg) = f(hgv) = [\tilde{f}(gv)](h) \]

So the following diagram commutes!

\[
\begin{array}{ccc}
V & \xrightarrow{\tilde{f}} & K[G] \\
| & | & | \\
g & & \rho(g) \\
\downarrow & & \downarrow \\
V & \xrightarrow{\tilde{f}} & K[G]
\end{array}
\]
Proof

Since matrix Jordan decomposition is compatible with commuting diagrams, the following diagram also commutes!

\[
\begin{array}{ccc}
V & \xrightarrow{\tilde{f}} & K[G] \\
\downarrow & & \downarrow \\
V & \xrightarrow{\tilde{f}} & K[G]
\end{array}
\]
Proof

• Using the two commuting diagrams, we get

\[ \tilde{\varphi}(a_s v) = \rho(g_s)[\tilde{\varphi}(v)] = \rho(g_s)[\tilde{\varphi}(v)] = \tilde{\varphi}(g_s v) \text{ [earlier diagram]} \]

• Again by the earlier diagram, \( \tilde{\varphi}(a_s v) = \rho(a_s)[\tilde{\varphi}(v)] \)

• As \( \tilde{\varphi} \) is injective, \( a_s v = g_s v \ \forall v \in V \)

• Hence \( a_s = g_s \)
Abstract Jordan decomposition behaves well wrt to group homomorphisms

Theorem

Let \( \phi : G \rightarrow H \) be an algebraic group map. Let \( g \in G \) and \( \phi(g) = h \). Then

1. \( \phi(g_s) = h_s \)
2. \( \phi(g_u) = h_u \)
Proof: Case \( \phi : G \to H \) is injective

- Identify \( G \) as a closed subgroup of \( H \). So \( K[G] = K[H]/I \)
- \( G = \{ h \in H | \rho_H(h)I = I \} \)
- Granting this, for \( h \in G \) (as above)

\[
\begin{array}{ccc}
K[H] & \to & K[H]/I \\
\downarrow \rho_H(h) & & \downarrow \rho_G(h) \\
K[H] & \to & K[G]
\end{array}
\]
Proof: Case $\phi : G \to H$ is injective

$$K[H] \xrightarrow{\pi} K[H]/I$$

![Diagram](Diagram.png)

- Why is $G = \{ h \in H | \rho_H(h)I = I \}$?
- If $g \in G, f \in I$, then $\rho_H(g)(f)(x) = f(xg) \forall x \in G$.
- Thus $\rho_H(g)(f)$ vanishes on all of $G$ and hence is in $I$.
- Conversely if $\rho_H(h)I = I$ and $f \in I$, then as $e \in G$,
  $$\rho_H(h)(f)(e) = f(h) = 0$$
- So $f(h) = 0 \forall f \in I$, then $h \in G$!
Proof: Case $\phi : G \rightarrow H$ is surjective

- So $K[H] \subseteq K[G]$
- $K[H]$ is $\rho_G(g)$ stable for all $g \in G$
- Granting this, for $g \in G$ (as above)

\[
\begin{array}{c}
K[H] \quad i \quad K[G] \\
\downarrow \quad \rho_H(g) \\
K[H] \quad i \quad K[G]
\end{array}
\]
Proof: Case $\phi: G \to H$ is surjective

- Why is $K[H]$ stable under $\rho_G(g)$ for all $g \in G$?
- Let $f \in K[H]$.
- Think of $f: H \to \mathbb{A}^1$
- $\tilde{f} = \rho_G(g)(f) \in K[G]$ and sends $x \sim f(xg)$
- Let $d$ be in kernel $\phi$.
- $\tilde{f}(dx_1) = f(dx_1g)$ and $\tilde{f}(x_1) = f(x_1g)$
- Want to show $f(dx_1g) = f(x_1g)$
- This is true as $f$ is a function on $H$!
What is a diagonalizable group?

$G/K$ is said to be diagonalizable if it is isomorphic to a closed subgroup of $D(n, K)$ for some $n$. 
Another description

Proposition

$G$ is diagonalizable if and only if $G$ is commutative and consists of semisimple elements

Proof.

$\implies$:

- Let $G \subseteq D(n, K) \subseteq GL(n, K)$
- $D(n, K)$ is commutative and therefore $G$ is.
- Let $G \xrightarrow{\phi} GL(n, K)$
- $g = g_s g_u$ implies $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$
- But $\phi(g) \in D(n, K)$ and hence is semisimple
- So $\phi(g)_u = 1$ and therefore $\phi(g_u) = 1$ and $g_u = 1$
- So all elements of $G$ are semisimple
Proposition

*G is diagonalizable if and only if G is commutative and consists of semisimple elements*

Proof.

\[ \Leftarrow : \]

- Let \( G \subseteq \text{GL}(n, K) \) be a closed subgroup
- Since \( G \) is commutative and all its elements are semisimple, they are simultaneously diagonalizable
- So \( G \) can be conjugated into \( \text{D}(n, K) \)
- Thus wlog let \( G \subseteq \text{D}(n, K) \)
- \( G \) is closed in \( \text{GL}(n, K) \) and hence closed in \( \text{D}(n, K) \)
Subgroups and homomorphic images ..

of diagonalizable groups are diagonalizable.

Use the alternate description for an immediate proof!
• Let $G$ be an algebraic group
• Let $X(G)$ be the **character group** of $G$
• $X(G) := \{ \xi : G \rightarrow G_m \}$
• It is an abelian group because $G_m$ is !
• $X(G) \subseteq K[G] = \{ f : G \rightarrow \mathbb{A}^1 \}$
• Recall that $K[G]$ is a $K$-vector space
**K-linear independence of characters**

**Lemma (Dedekind?)**

Let $G$ be an abstract group and let $X = \{ f : G \to K^* \}$ be the group of its characters. Then $X$ is a $K$-linear independent subset of $S = \{ f : G \to K \}$, the $K$-space of $K$ valued functions of $G$.

**Proof:**

- Let $\xi_1, \ldots, \xi_n$ be distinct characters of $G$.
- If possible, let $\sum a_i \xi_i = 0$ for scalars $a_i \in K$ not all 0.
- Assume $n$ is minimum.
- $n \neq 1$.
- This is because $a_1 \xi_1 = 0$ implies $a_1 = 0$ as $\xi_1 : G \to K^*$.
- Thus wlog assume $a_1, a_2 \neq 0$.
- Find $g \in G$ such that $\xi_1(g) \neq \xi_2(g)$. 

$K$-linear independence of characters

- $\sum a_i \xi_i(x) = 0$ for each $x \in G$ - (*)
- Multiply (*) by $\xi_1(g)$ to get
  
  $$a_1 \xi_1(x) \xi_1(g) + a_2 \xi_2(x) \xi_1(g) + \ldots + a_n \xi_n(x) \xi_1(g) = 0$$

- We also have $\sum a_i \xi_i(xg) = 0$. That is,
  
  $$a_1 \xi_1(x) \xi_1(g) + a_2 \xi_2(x) \xi_2(g) + \ldots + a_n \xi_n(x) \xi_n(g) = 0$$

- Subtract (2) from (1) to get
  
  $$a_2 \xi_2(x)(\xi_1(g) - \xi_2(g)) + \ldots + a_n \xi_n(x)(\xi_1(g) - \xi_n(g)) = 0$$

- Nontrivial smaller relation as $a_2 \neq 0$ and $\xi_1(g) - \xi_2(g) \neq 0$.
- $n$ is not minimum!
\textit{d}-groups

- $G/K$ is a \textit{d-group} if $X(G) \subseteq K[G]$ generates $K[G]$ as a $K$ vector space.
- Thanks to Dedekind's Lemma, thus $G/K$ is a \textit{d-group} if and only if $X(G)$ is a $K$-basis of $K[G]$. 
$D(n, K)$ is a d-group!

- Let $G = D(n, K)$
- Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$
- Define character $\phi_{\bar{a}} : G \rightarrow \mathbb{G}_m$ sending the diagonal $i^{th}$ entry to its $a_i^{th}$ power.
- The corresponding $K$-algebra map is

$$\phi_{\bar{a}}^* : K[t, t^{-1}] \rightarrow K[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$$

$$t \sim t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n}$$

- $\phi_{\bar{a}}^*$ generate $K[G]$ as a $K$ vector space!
- These are the only characters by linear independence of characters!
- $X(D(n, K)) \sim \mathbb{Z}^n$
Character group of $\mathbb{G}_m$ by hand

- Let $G = \mathbb{G}_m$ and let $a \in \mathbb{Z}$
- As before define character $\phi_a : G \to \mathbb{G}_m$ sending $x \mapsto x^a$
- The corresponding $K$-algebra map is

$$\phi_a^* : K[t, t^{-1}] \to K[t_1^{\pm 1}]$$

$$t \mapsto t_1^a$$

- There are no other characters!
- If $f : K[t, t^{-1}] \to K[t_1^{\pm 1}]$ is a character, then

$$t \mapsto f(t)$$

$$t^{-1} \mapsto f(t)^{-1}$$

- So $f(t) = \alpha t^n$ for $\alpha \in K$
Character group of $G_m$ by hand

- If $f(t) = \alpha t^n$, what is the map induced from $K^* \to K^*$?
- It is $x \mapsto \alpha x^n$ for $x \in K^*$
- This is a group map implies $\alpha^2 = \alpha$
- So $\alpha = 0$ or $\alpha = 1$
- But $\alpha$ cannot be $0$ for $f(t^{-1}) = (\alpha t^n)^{-1}$.
- Thus $X(G_m) \simeq \mathbb{Z}$. 
Closed subgroups of $d$-groups

Proposition

Let $H \hookrightarrow G$ be a closed subgroup of $d$-group $G$. Then $H$ is a $d$-group.

Proof.

• Thus $A = K[G]$ and $K[H] = A/I$ where $I$ is ideal defining $H$
• $\text{Res} : K[G] \to K[H]$ sends $f : G \to K$ to $f|_H : H \to K$ and is $K$-linear and surjective.
• The restriction of a character of $G$ is a character of $H$!
• Thus $\text{Res}(X(G)) \subseteq X(H)$
• $X(G)$ generates $K[G]$ as a $K$-space
• Thus $\text{Res}(X(G))$ generates $\text{Res}(K[G]) = K[H]$ as a $K$ space.
• Thus $X(H)$ generates $K[H]$ and so $H$ is a $d$-group.
**Proposition**

*G is diagonalizable iff it is a d-group*

**Proof.**

⇒

- Let $G$ be diagonalizable
- So it is a closed subgroup of $D(n, K)$
- But $D(n, K)$ is a $d$-group
- And closed subgroups of $d$-groups are again $d$-groups
- So $G$ is a $d$-group
Proposition

$G$ is diagonalizable iff it is a $d$-group

proof $\iff$

- Let $G$ be a $d$-group
- So $X(G)$ is a $K$-basis of $K[G]$
- Pick characters $\xi_1, \xi_2, \ldots, \xi_n$ which generate $K[G]$ as a $K$-algebra
- You only need finitely many as $K[G]$ is a f.g $K$-algebra
- Define $\phi : G \to D(n, K)$ sending $g$ to the diagonal matrix $(\xi_i(g))$
- This is an algebraic group map!
**d for diagonalizable**

**Proposition**

*G is diagonalizable iff it is a d-group*

**proof** $\iff$

- Kernel $\phi$ is trivial
- Because if not $\xi_i(g) = \xi_i(h) \forall i$
- Thus $f(g) = f(h) \forall f \in K[G]$ as $\{\xi_i\}$ generate $K[G]$ as a $K$-algebra
- Thus $G$ is commutative and consists of diagonalizable elements
- So $G$ is diagonalizable!
- Caution: $\phi$ needn't be a closed immersion even though it is injective on $\overline{k}$ points
- So (I guess) we are not claiming $G$ is isomorphic to its image, a closed subgroup of this $D(n, K)$