Algebraic groups over $K = \bar{K}$ (Lecture VI)
Derivations, Lie algebras of algebraic groups, Examples

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What is a derivation

- Let $F$ be a field and $A$, an $F$-algebra
- Let $M$ be an $A$ module
- An $F$-derivation is an $F$ linear map $D : A \to M$ such that

$$D(ab) = aD(b) + bD(a).$$

- $F$-linearity forces that $D(f) = 0$ for each $f \in F$
- This is because on the one hand

$$D(f) = D(f.1) = fD(1) + D(f)$$

- But on the other hand $D(f.1) = fD(1)$.
- The set of $F$-derivations $Der_F(A, M)$ is an $A$ module
- If $a \in A$ and $D$ is an $F$-derivation, then $aD : A \to M$ sends $b \mapsto aD(b)$. 
Universal derivation

- Let $A$ and $F$ be as before
- There exists an $A$-module $\Omega_{A/F}$ and a derivation $d : A \to \Omega_{A/F}$ such that for every $A$-module $M$

$$\text{Hom}_{A-mod} (\Omega_{A/F}, M) \to \text{Der}_F (A, M)$$

\[ f \leftrightarrow f \circ d \]

is a bijection.

- We suppress the $F$ in the notation and simply call the module $\Omega_A$
- $d : A \to \Omega_A$ is called the universal derivation
To say it with a picture

\[ A \xrightarrow{d} \Omega_A \xrightarrow{D} M \]

\[ \exists! f \]

Derivations  |  Lie algebras  |  Identifications  |  Lie brackets  |  Examples
When \( A \) is a field

- Let \( A/F \) be a field extension
- If \( A/F \) is a finite separable extension, then \( \Omega_{A/F} = 0 \)
- If \( A/F \) is a purely transcendental extension, then the dimension of \( \Omega_{A/F} \) is the transcendence degree of \( A/F \)
- If \( A/F \) contains a sub-extension \( L/F \) such that \( A/L \) is finite separable and \( L/F \) is purely transcendental, then the dimension of \( \Omega_{A/F} \) is the transcendence degree of \( A/F \)
Recipe for $\Omega_A$

- Let $A = F [x_1, x_2, \ldots x_n] / I$ where $I$ is an ideal.
- Let $I$ be generated by polynomials $\{f_j\}_{j \in J}$.
- Let $X$ be the free $A$ module on the set of symbols $\{dx_1, dx_2, \ldots x_n\}$.

$$X = \bigoplus_{i=1}^{n} A dx_i$$

- Let $T_j$ be the polynomial $\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} dx_i$.
- Then $\Omega_A$ is the $A$-module

$$\Omega_A := X / (T_j)_{j \in J}$$
Examples

• If $X = \text{Spec } A$ is an affine variety, we call $\Omega_A$ sometimes as $\Omega_X$
• Let $X = \mathbb{A}_k^n$ be the affine $n$-space
• Then $\Omega_X \cong \bigoplus A^n$
• Notice $\text{dim } X = \text{rank } \Omega_X$

• Let $Y$ be the unit circle over a characteristic not 2 field $k$.
• So $Y = \text{Spec } k[x, y](x^2 + y^2 - 1) = \text{Spec } A$
• So $\Omega_Y = \frac{Adx \oplus Ady}{(2xdx + 2ydy)}$
• Exercise : Check that $\Omega_Y$ is in fact $\cong A$ !
• Hint : $z = xdy - ydx$ generates $\Omega_Y$ as an $A$-module.
  Why is it torsion free?
• Notice $\text{dim } Y = \text{rank } \Omega_Y$
Dimension $\equiv$ rank

- Back to $K = \overline{K}$.
- Look at a connected algebraic group $G/K$
- Note that by definition for us, the coordinate ring $K[G]$ is reduced
- Since $G$ is connected, it is irreducible and $K[G]$ is a domain
- Let $K(G)$ denote the function field of $G$
- Then in fact $\dim G = \text{rank } \Omega_G$!
Proof: Dimension = rank

- Rank of $\Omega_{K[G]}$ is the $K(G)$-dimension of $\Omega_{K[G]} \otimes_{K[G]} k(G)$
- Set $S = K[G] \setminus \{0\}$.
- Then

$$\Omega_{K[G]} \otimes_{K[G]} k(G) = \Omega_{K[G]} \otimes_{K[G]} S^{-1}K[G]$$
$$= \Omega S^{-1}K[G]$$
$$= \Omega_{K(G)}$$

- Thus rank of $\Omega_{K[G]}$ is the $K(G)$-dimension of $\Omega_{K(G)}$
- But the latter is the transcendence degree of $K(G)/K$ which is just $\dim G$
Exercise caution when not reduced

- Revisit the example of the circle over char 2 fields
- So \( Y = \text{Spec } k[x, y](x^2 + y^2 - 1) = \text{Spec } A \)
- \( A \) is not reduced!

\[
\Omega_Y = \frac{Adx \oplus Ady}{(2xdx + 2ydy)} \cong A \oplus A
\]

- Thus rank of \( \Omega_Y \) is 2!
What is a Lie algebra

- Let \( k \) be a field
- Let \( \mathcal{L} \) be a (finite dimensional) vector space over \( k \)
- Let \([,] : \mathcal{L} \times \mathcal{L} \to k\) be a \( k \)-bilinear form such that

\[
\text{Alternating : } [x, x] = 0 \forall x \in \mathcal{L} \\
\text{Jacobi identity : } [x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0.
\]

- \( \mathcal{L} \) along with \([,]\) is called a lie algebra over \( k \).
- Since \([x + y, x + y] = 0\), we see \([x, y] = -[y, x]\).
An example

- Let $A/K$ be a (finite dimensional) associative algebra over $K$
- Let $[a, b] = ab - ba$ for every $a, b \in A$
- This makes $A$ into a Lie algebra
Let $A/K$ be a (finite dimensional) associative algebra over $k$

Let $X = \text{End}_K(A, A)$ be a Lie algebra under $[x, y] = xy - yx$ for every $x, y \in X$

Let $\mathcal{D} = \text{Der}_K(A, A) \subseteq X$

For $d, e \in \mathcal{D}$, the Lie bracket $[d, e] \in \mathcal{D}$ (Verify!)

Thus $\mathcal{D}$, with the induced Lie bracket is a Lie algebra over $K$
We are given a linear algebraic group $G/K$ where $K = \overline{K}$.

We would like to associate to it a Lie algebra \textit{functorially}.

Then $\mathcal{D} = \text{Der}_K(A, A)$ where $A = K[G]$ is a candidate.

But $\mathcal{D}$ is not a finite dimensional $K$-vspace!

We will define the Lie algebra of $G$ to be a fd $K$-subspace of $\mathcal{D}$. 
\[ D_1 := \text{Lie}(G) \]

The first interpretation

- Recall the dual representation of left translation

\[ \lambda : G \times A \to A \]

\[ g, f \mapsto [x \mapsto f(g^{-1}x)] \]

- We call \( \lambda_g : A \to A \) to be the one sending \( f \mapsto \lambda(g, f) \).
- \( d \in D \) is said to be left invariant if

\[ d \circ \lambda_g = \lambda_g \circ d \quad \forall g \in G \]

- It is a Lie-sub algebra of \( D \) (Check!) ..
- .. which we will call \( D_1 := \text{Lie}(G) \)
- At this point it is not clear why \( \text{Lie}(G) \) is finite dimensional \( K \)-space
- But we will establish an isomorphism with a clearly fd space soon.
$D_2$, the point derivations

The second interpretation

• Since $G$ is an algebraic group, it contains the identity $e$
• This is the map $e : \text{Spec } K \to G$
• Which corresponds to $e^* : K[G] \to K$ which sends $f \mapsto f(e)$
• Setting $A = K[G]$ again, $e^*$ is just the counit $\epsilon : A \to K$
• Thus $K$ is an $A$ module via $\epsilon$
• Look at $D_2 := \text{Der}_K(A, k)$ where $K$ is an $A$-module via $\epsilon$.
• This is a $K$-vector space.
• We will define the Lie bracket on it later.
\( \mathcal{D}_3 \), the tangent space

The third interpretation

- Since \( G \) is an algebraic group, it contains the identity \( e \)
- Let \( I \) be the maximal ideal defining \( e \) in \( A = K[G] \)
- Thus \( 0 \to I \to A \xrightarrow{\epsilon} K \to 0 \) is exact.
- Then \( \mathcal{D}_3 := \text{Hom}_K \left( I/I^2, K \right) = (I/I^2)^* \)
- This is the tangent space definition!
- Thus \( \mathcal{D}_3 \) is a finite dimensional vector space over \( K \).
- Again we will define the Lie bracket on it later.
The ring $K[t]/(t^2) = K[\tau]$ is called the ring of dual numbers.

We have a $k$-map, $\gamma : K[\tau] \to K$ which sends $\tau \sim 0$

Thus we have an induced map $G(\gamma) : G(K[\tau]) \to G(k)$ which sends $A \xrightarrow{f} K[\tau]$ to $A \xrightarrow{f} K[\tau] \xrightarrow{\gamma} K$

Define $\mathcal{D}_4$ to be the kernel of $G(\gamma)$, namely

$$1 \to \mathcal{D}_4 \to G(K[\tau]) \xrightarrow{G(\gamma)} G(K) \to 1$$

Thus $\mathcal{D}_4$ is apriori just a subset of $G(K[\tau])$

We will define the vector space structure and a Lie bracket on it later.
\( \mathcal{D}_4 = \mathcal{D}_2 \)

- Let \( f \in D_4 \)
- Since first of all \( f \in G(K[\tau]) \), it is a \( K \)-alg map \( f : A \rightarrow K[\tau] \)
- Since \( K[\tau] = K \oplus K\tau \), let \( f = f_1 + \tau f_2 \) where \( f_i : A \rightarrow K \).
- Recall that \( G(\gamma) : G(K[\tau]) \rightarrow G(K) \) sends \( A \xrightarrow{f} K[\tau] \) to \( A \xrightarrow{f} K[\tau] \xrightarrow{\gamma} K \)
- So we have \( \gamma \circ f = \epsilon \)
- So \( f_1 = \epsilon \)
- Each implication above is really an iff statement
- So \( f \in \mathcal{D}_4 \) if and only if \( f = \epsilon + \tau f_2 \)
$\mathcal{D}_4 = \mathcal{D}_2$

- $f_2 : A \rightarrow K$ is in $\mathcal{D}_2$! Here’s a proof:
- Recall $f(ab) = f(a)f(b)$ as $f$ is a $K$-algebra map. Also $\epsilon$.
- Since $f = \epsilon + \tau f_2$, we have

$$\epsilon(ab) + \tau f_2(ab) = (\epsilon(a) + \tau f_2(a)) (\epsilon(b) + \tau f_2(b))$$

- Multiply this out and use $\tau^2 = 0$ and $\epsilon(ab) = \epsilon(a)\epsilon(b)$
- So we have $f_2(ab) = f_2(a)\epsilon(b) + \epsilon(a)f_2(b)$
- Hence $f_2 \in \mathcal{D}_2$
- And now by identifying $f \in \mathcal{D}_4$ with $f_2 \in \mathcal{D}_2$, we can see this is a set bijection.
Let $d \in D_2$

Recall the exact sequence $0 \to I \to A \xrightarrow{\epsilon} K \to 0$

We claim that $d(I^2) = 0$

This is because for $i, j \in I$, we have

$$d(ij) = \epsilon(j)d(i) + \epsilon(i)d(j) = 0$$

Thus $d|_I : I \to k$ factors through $\tilde{d} : I/I^2 \to K$

$\tilde{d} \in D_3$!
Conversely, let \( f \in \mathcal{D}_3 \)

- We will manufacture \( \hat{f} : A \rightarrow K \in \mathcal{D}_2 \)
- Note that \( 0 \rightarrow I \rightarrow A \xrightarrow{\epsilon} K \rightarrow 0 \) splits as vector spaces
- Thus \( A = I \oplus K \)
- Define \( \hat{f}(i + \lambda) = f(i) \) for \( i \in I \) and \( \lambda \in K \).
- It is an easy check that \( \hat{f} \in \mathcal{D}_2 \)!

For \( i, j \in I \) and \( c, e \in K \), we have

\[
\hat{f}((i + c)(j + e)) = \hat{f}(ij + cj + ie + ce) \\
= f(ij) + f(cj) + f(ie) \\
= 0 + cf(j) + ef(i) \\
= \epsilon(c)f(j) + \epsilon(e)f(i) \\
= \epsilon(i + c)f(j) + \epsilon(j + e)f(i) \\
= \epsilon(i + c)\hat{f}(j + e) + \epsilon(j + e)\hat{f}(i + c)
\]
• Identification is best done using the Hopf algebras language
• We shall give the recipe to go from $\mathcal{D}_1$ to $\mathcal{D}_2$ sans explanation
• Let $D \in \mathcal{D}_1$. Thus $D : A \to A$
• Composing with the co-unit $\epsilon$ gives an element $\epsilon \circ D$ in $\mathcal{D}_2$.

$A \xrightarrow{D} A \xrightarrow{\epsilon} K$
We shall give the recipe to go from $\mathcal{D}_2$ to $\mathcal{D}_1$ sans explanation.

Let $d \in \mathcal{D}_2$. Thus $f : A \rightarrow K$.

Recall the co-multiplication map $\mu^* : A \rightarrow A \otimes_K A$.

Form the following composition $\tilde{d}$

$$A \xrightarrow{\mu^*} A \otimes A \xrightarrow{\text{id} \otimes d} A \otimes K \simeq A$$

$\tilde{d} \in \mathcal{D}_1$!
Dimension of $\text{Lie}(G)$

- Dimension of $\text{Lie}(G) =$ dimension of the tangent space at $e$
- In general, tangent space dimension $\geq$ dimension of $G$
- For smooth points, there is an equality
- Thus dimension $\text{Lie}(G)$ as a vspace over $K$ is just $\dim G$. 
Remarks about $\mathcal{D}_4$

- Let $A \xrightarrow{f} K[\tau] \in \mathcal{D}_4 \subseteq G(K[\tau])$
- As before, $f = \epsilon + \tau f_2$
- View $\epsilon : A \rightarrow K$ as picking out the identity point of $G$
- View $f_2$ (which we can identity to be in $\mathcal{D}_2$ and hence in $\mathcal{D}_3$) as picking out a tangent vector at $e_G$!
• Define $[M, N] = MN - NM$ for $M, N \in \text{End}_K(A)$
• $\mathcal{D}_1 \subseteq \text{End}_K(A)$ becomes a Lie subalgebra under this Lie bracket.
• We will transport this Lie structure to $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$!
Let $d_1, d_2 \in D_2$

Then $[d_1, d_2]$ is given by the composition

$$A \xrightarrow{\mu^*} A \otimes A \xrightarrow{d_1 \otimes d_2 - d_2 \otimes d_1} K \otimes K \cong K$$
Vector space structure on $\mathcal{D}_4$

- For $f, g \in \mathcal{D}_4 \subseteq G(K[\tau])$, set $f + g := f \ast g$ where $\ast$ is the group operation in $G(K[\tau])$
- It turns out that $f \ast g \in \mathcal{D}_4$
- And also $\ast$ restricted to $\mathcal{D}_4$ is abelian though it might not be abelian on all of $G(K[\tau])$
- The $K$-scalar multiplication is a little more involved to define
- For $\lambda \in K$, look at the $K$-algebra map

$$\gamma_\lambda : K[\tau] \rightarrow K, \quad \tau \mapsto \lambda \tau$$

- This induces a group map $G(\gamma_\lambda) : G(K[\tau]) \rightarrow G(K[\tau])$.
- For $f \in \mathcal{D}_4$, set $\lambda f := G(\gamma_\lambda)(f)$. 
Let $f = \epsilon + \tau d_1$ and $f' = \epsilon + \tau d_2$ be in $\mathcal{D}_4$.

Then we would like to define $[f, f']$.

Let $R = K[u, v]$ where $u^2 = v^2 = 0$.

Let $\gamma_u : K[\tau] \to R$ send $\tau \sim u$

Let $\gamma_v : K[\tau] \to R$ send $\tau \sim v$

Let $\gamma_{uv} : K[\tau] \to R$ send $\tau \sim uv$

Thus we have $G(\gamma_u) : G(K[\tau]) \to G(R)$ and similarly $G(\gamma_v), G(\gamma_{uv})$. 

[ [ , ] on $\mathcal{D}_4$ ]
Define $g_1 := G(\gamma_u)(f) = \epsilon + ud_1$

Define $g_2 := G(\gamma_v)(f) = \epsilon + vd_2$

Let $g = g_1 g_2 g_1^{-1} g_2^{-1} \in G(R)$

Look at $G(\gamma_{uv})^{-1}(g)$

It turns out to be in $\mathcal{D}_4$ and we set it to be $[f, f']$

More explicitly, $g = \epsilon + (uv)[d_1, d_2]$

And $[f, f'] = \epsilon + \tau[d_1, d_2] \in D_4$
Let $G = \mathbb{G}_a$. Thus $A = K[t]$

Since $G$ is smooth and one dimensional, $\text{Lie}(G) \cong K$

We can also use the $\mathcal{D}_2$ interpretation to see this

Derivations $d : K[t] \to K$ are determined by $d(t) \in K$.

What is the Lie bracket on $\text{Lie}(G)$?

Let $d_1, d_2 \in \mathcal{D}_2$. Then $d = [d_1, d_2]$ is given by the composition

$$A \xrightarrow{\mu^*} A \otimes A \xrightarrow{d_1 \otimes d_2 - d_2 \otimes d_1} K \otimes K \cong K$$

But $\mu^*(t) = 1 \otimes t + t \otimes 1$

Then we get

$$d(t) = d_1(1)d_2(t) + d_1(t)d_2(1) - d_2(1)d_1(t) - d_2(t)d_1(1) = 0$$

This is because $d_i(1) = 0$

Thus $\text{Lie}(G) = K$ and $[a, b] = \forall a, b \in \text{Lie}(G)$
Let $G = \mathbb{G}_m$. Thus $A = K[t, t^{-1}]$

• As before, it is easy to see $\text{Lie}(G) \simeq K$.

• What is the Lie bracket on $\text{Lie}(G)$?

• Let $d_1, d_2 \in \mathcal{D}_2$. Then $d = [d_1, d_2]$ is given by the composition

$$A \xrightarrow{\mu^*} A \otimes A \xrightarrow{d_1 \otimes d_2 - d_2 \otimes d_1} K \otimes K \simeq K$$

• But $\mu^*(t) = t \otimes t^{-1}$

• Then we get $d(t) = d_1(t)d_2(t) - d_2(t)d_1(t) = 0$

• Thus $\text{Lie}(G) = K$ and $[a, b] = \forall a, b \in \text{Lie}(G)$