Algebraic groups over $K = \overline{K}$ (Lecture X)

$G/B$, Weyl groups, Cartans in Borels, Regular and singular tori

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Spring 2016
$G/B$ versus the collection of Borels

- Let $G$ be a connected algebraic group through out this lecture.
- Let $B$ be a Borel subgroup
- Let $\mathbb{B}$ denote the collection of all Borels of $G$
- Then define the correspondence $\phi : \mathbb{B} \rightarrow G/B$ which sends
  \[ xBx^{-1} \sim xB \]

- Since Borels are conjugate, $\phi$ is defined on $\mathbb{B}$ and clearly onto
- It is well defined because of the normalizer theorem!
- Because if $yBy^{-1} = xBx^{-1}$, then $x^{-1}y \in N_G(B) = B$ and hence $y \in xB$ and hence $yB \subseteq xB$ etc.
- Similarly it is easy to check $\phi$ is 1–1.
G action on B and G/B

- G acts on B by

\[ g, B' \leadsto gB'g^{-1} \]

- G acts on G/B by

\[ g, xB \leadsto gxB \]

- The action of G on B and G/B is compatible with the correspondence \( \phi \)

\[
\begin{align*}
xBx^{-1} & \xrightarrow{\phi} xB \\
g & \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow g \\
gxBx^{-1}g^{-1} & \xrightarrow{\phi} gxB
\end{align*}
\]
Fixed points of $G/B$ under $H$

- Let $H$ be a subgroup of $G$
- Let $B^H$ denote the Borels containing $H$
- Let $(G/B)^H$ denote the fixed points under $H$ of $G/B$
- Claim: $\phi$ induces a bijection between $B^H$ and $(G/B)^H$
- To see this, let $H \subseteq B'$ say.
  - If $h \in B'$, then $h(B')h^{-1} = B'$ and hence $h(\phi(B')) = \phi(B')$
  - Conversely, let $hxB = xB$ for some coset $xB$ and for all $h \in H$
    - Then $h(xBx^{-1})h^{-1} = h\phi^{-1}(xB)h^{-1} = \phi^{-1}(xB) = xBx^{-1}$
    - So $h \in N_G(xBx^{-1})$ and hence by normalizer theorem, $h \in xBx^{-1}$
Let $S$ be a torus in $G$

Then by rigidity of $S$, we have seen $N_G(S)^\circ = C_G(S)^\circ$

But by connectedness theorem, $C_G(S) = C_G(S)^{circ}$

So $\frac{N_G(S)}{C_G(S)} = \frac{N_G(S)}{N_G(S)^\circ}$ which is finite!

Set this abstract finite group to be $W(G, S)$, the Weyl group of $G$ with respect to $S$
Weyl group of $G$

- Let $T$ be a maximal torus in $G$ and let $T'$ be another maximal torus
- Then we know $T' = gTg^{-1}$
- Hence $N_G(T') = gN_G(T)g^{-1}$ and $C_G(T') = gC_G(T)g^{-1}$
- So $W(G, T) \cong W(G, T')$
- We call this to be the Weyl group of $G$
Borels containing a Cartan

- Let $T$ be a maximal torus of $G$
- Let $C = C_G(T)$ be the associated Cartan
- Clearly $T \subseteq C$ and hence if $B \in \mathbb{B}^C$, then $B \in \mathbb{B}^T$
- Claim: In fact, $\mathbb{B}^C = \mathbb{B}^T$!
- Note that $C$ is connected and nilpotent (because $T$ is the unique maximal torus of $C$)
- Thus $C$ is connected and solvable and lies in some Borel $B$
- As observed before, $C \in \mathbb{B}^C \subseteq \mathbb{B}^T$ therefore also.
- If $B' = xBx^{-1} \in \mathbb{B}^T$, then $T$ and $xTx^{-1}$ are maximal tori of $B'$
- So there exists $y \in B'$ such that $yxTx^{-1}y^{-1} = T$
- So $yxCx^{-1}y^{-1} = yxC_G(T)x^{-1}y^{-1} = C_G(T) = C$
- So $C = yxCx^{-1}y^{-1} \subseteq y(xBx^{-1})y^{-1} = yB'y^{-1} = B'$ as $y \in B'$
- Hence $B' \in \mathbb{B}^C$
Action of $C_G(T)$ on $B^T$

- Let $T$ be a maximal torus of $G$
- Let $C = C_G(T)$ be the associated Cartan
- We have seen $B^C = B^T$
- So
- Thus $C$ is connected and solvable and lies in some Borel $B$
- As observed before, $C \in B^C \subseteq B^T$ therefore also.
- If $B' = xBx^{-1} \in B^T$, then $T$ and $xTx^{-1}$ are maximal tori of $B'$
- So there exists $y \in B'$ such that $yxTx^{-1}y^{-1} = T$
- So $yxCx^{-1}y^{-1} = yxC_G(T)x^{-1}y^{-1} = C_G(T) = C$
- So $C = yxCx^{-1}y^{-1} \subseteq y(xBx^{-1})y^{-1} = yB'y^{-1} = B'$ as $y \in B'$
- Hence $B' \in B^C$
Action of $W(G, T)$ on $\mathbb{B}^T$

- Let $T$ be a maximal torus of $G$
- $W(G, T) = N_G(T)/C_G(T)$
- Recall that $G \times \mathbb{B} \to \mathbb{B}$ by sending $g, B \sim gBg^{-1}$
- In particular, if $B \supseteq T$ and $g \in N_G(T)$, then $gBg^{-1} \supseteq gTg^{-1} = T$
- Hence we get an induced action $N_G(T) \times \mathbb{B}^T \to \mathbb{B}^T$
- We have already observed that $C_G(T) \subseteq B$ whenever $B \in \mathbb{B}^T$
- Thus $C_G(T)$ acts on $\mathbb{B}^T$ trivially!
- So we get an action $W(G, T) \times \mathbb{B}^T \to \mathbb{B}^T$
Regular and singular tori

- Let $S$ be a torus of $G$
- If $B^S$ is finite, we call $S$ regular
- Otherwise, we call it singular
- We will try to see some examples and criterion of regularity and singularity of tori in $G$. 
Maximal tori are regular

**Theorem**

Let $T$ be a maximal torus. Then $W(G, T)$ acts simply transitively on $\mathcal{B}^T$. Hence $|\mathcal{B}^T| = |W(G, T)| < \infty$ and $T$ is regular.
Proof: Transitive action

- Let \( B_1, B_2 \in \mathcal{B}^T \).
- Then \( xB_2x^{-1} = B_1 \) for some \( x \in G \).
- Then \( xTx^{-1} \) and \( T \) are maximal tori in \( B_1 \).
- So there is \( y \in B_1 \) such that \( yxTx^{-1}y^{-1} = T \).
- So \( yx \in N_G(T) \).
- But \( yxB_2x^{-1}y^{-1} = yB_1y^{-1} = B_1 \).
- So \( [yx] \in N_G(T)/C_G(T) \) sends \( B_2 \sim B_1 \).
Proof: Free action

- Let $B \in \mathbb{B}^T$
- We want to show $stab_{W(G,T)}(B) = \{1\}$
- So if $x \in N_G(T)$ such that $xBx^{-1} = B$, we want to show $x \in C_G(T)$
- $xBx^{-1} = B$ implies $x \in N_G(B) = B$
- So the problem reduces to thinking about a connected solvable group
- FACT: In fact, $N_B(T) = C_B(T)$
- Granting this fact, we get that $x \in B$ and in $N_G(T)$, so $x \in N_B(T) = C_B(T)$
- Hence $x \in C_G(T)$
Borels of centralizers of tori

• Let $B$ be a Borel containing $S$, a torus.
• Let $C = C_G(S)$
• Let $X = (G/B)^S$ and $Y$ be any irreducible component of $X$
• Theorem: $C$ acts transitively on $Y$
• Granting this, thus $Y \cong C/C \cap B$
• Since $Y$ is closed subset of a complete space $(G/B)^S$, it is complete too
• But it is also quasi-projective, so $Y$ is projective
• Thus $C \cap B = C_B(S)$ is a parabolic of $C$ and is connected
• Since $B$ is solvable, so is any subgroup of it
• Hence $C_B(S)$ is a Borel of $C$
Criterion I for regularity

Theorem

Let $S$ be a torus of $G$ and $C = C_G(S)$. Then $S$ is regular iff $C$ is solvable.
Proof of Criterion I for regularity

• Let $B$ be a Borel containing $S$ and $C = C_G(S)$
• Let $X = (G/B)^S$ and $Y$ be any irreducible component of $X$
• Have seen $Y \cong C/C_B(S)$ and $C_B(S)$ is a Borel of $C$.
• Thus dim $Y$ is the codimension of a Borel of $C$ in $C$.
• So every irreducible component of $X$ has same dimension.
• $S$ regular iff $\mathcal{B}^S$ is finite iff $|(G/B)^S|$ is finite
• Thus $S$ regular iff $Y$ has dimension 0 iff $C$ is its own Borel
• Thus $S$ regular iff $C$ is solvable
Defining $I(T)$

- Let $T$ be a maximal torus of $G$
- Define $I(T)$ to be

$$I(T) = \left( \bigcap_{B \in \mathcal{B}^T} B \right)^\circ$$

- Since $C_G(T) \subseteq B$ for every $B \in \mathcal{B}^T$ and $C_G(T)$ is connected, we have

$$C_G(T) \subseteq I(T)$$
Defining $\phi$

- Recall $Ad : G \to GL(V)$ where $Lie(G) = V$
- $Ad|_T$ gives a decomposition $V = V_0 \bigoplus \bigoplus_{\alpha \in \phi} V_\alpha$ where $\phi$ is a finite subset of $X(T) \setminus \{1\}$
- $V_0 = \{v \in Lie(G) | Ad(t)v = v \forall t \in T\}$
- Note that for any $t \in T$, we have $Int(t) : C_G(T) \to C_G(T)$ is the identity map
- Thus $Ad|_T : Lie(C_G(T)) \to Lie(C_G(T))$ is the trivial map.
- Hence $Lie(C_G(T)) \subseteq V_0$
- **FACT** : $Lie(C_G(T)) = V_0$
- Granting this ...
Defining $\psi$

- Note that for any $t \in I(T)$, we have $\text{Int}(t) : I(T) \to I(T)$
- Thus $\text{Ad}|_T : \text{Lie}(I(T)) \to \text{Lie}(I(T))$ where $\text{Lie}(I(T))$ is a Lie-sub algebra of $\text{Lie}(G)$
- $C_G(T) \subseteq I(T)$
- So $V_0 = \text{Lie}(C_G(T)) \subseteq \text{Lie}(I(T))$
- Thus $V = \text{Lie}(I(T)) \oplus \bigoplus_{\alpha \in \psi} V_\alpha$ where $\psi \subseteq \phi$
Defining $T_\alpha$

- Let $\alpha : T \rightarrow \mathbb{G}_m$ for $\alpha \in \psi$
- Since $\{1\} \not\in \psi$ (or $\phi$), $\alpha$ is not the trivial character
- Set $T_\alpha$ to be $(\ker \alpha)^\circ$
- Since $\operatorname{Img} \alpha$ is closed and irreducible and $\mathbb{G}_m$ is one dimensional, $T_\alpha$ is a codimension one torus of $T$
Criterion II for regularity

Theorem

Let $S$ be a torus of $G$ and let $T$ be a maximal torus containing $G$. Then $S$ is singular iff $S \subseteq T_\alpha$ for some $\alpha \in \psi$. 

Example of singular tori

Fixing $T$ and $G$

- Let $T = D(2, K) \subseteq G = \text{GL}(2, K)$
- $X(T) = \mathbb{Z} \times \mathbb{Z}$
- For $m, n \in \mathbb{Z}$, the character $\alpha_{m,n} \in X(T)$ corresponds to the map $T \to \mathbb{G}_m$ sending

  $$
  \begin{pmatrix}
  t_1 & 0 \\
  0 & t_2
  \end{pmatrix}
  \sim t_1^m t_2^n
  $$

- $\text{Lie}(G) = M_2(K)$
Example of singular tori

The adjoint representation and $\psi$

- $\text{Ad} : T \to [M_2(K) \to M_2(K)]$ sends

  \[
  \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \sim [M \to \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} M \begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_2^{-1} \end{pmatrix}]
  \]

- A non trivial character $\alpha_{m,n} \in X(T) \in \phi$ iff there exists non-trivial $M \in M_2(K)$ such that for all $\bar{t} \in T$,

  \[
  \text{Ad}(\bar{t})(M) = \alpha_{m,n}(\bar{t})M
  \]
Example of singular tori

The adjoint representation and $\psi$

- That is $\alpha_{m,n} \in \phi$ iff we can find $a, b, c, d \in K$ not all 0 such that for all $t_1, t_2 \in K^*$, we have

$\begin{pmatrix}
a & t_1 t_2^{-1} b \\
t_2 t_1^{-1} c & d
\end{pmatrix} = \begin{pmatrix}
t_1^m t_2^n a & t_1^m t_2^n b \\
t_1^m t_2^n c & t_1^m t_2^n d
\end{pmatrix}$

- Solving we get $\alpha_{1,-1}$ and $\alpha_{-1,1}$ are the only roots and the corresponding eigen (matrices) are $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- FACT: For reductive groups, $\phi = \psi$

- Granting this..
Example of singular tori

Finding $T_\alpha$

- The kernel of $\alpha_{1,-1}$ and the kernel of $\alpha_{-1,1}$ are the same and equal to $\mathbb{G}_m$ which is $
\begin{pmatrix}
t & 0 \\
0 & t
\end{pmatrix}$
for all $t \in K^*$. 

- This is in $Z(G)^\circ$ and hence contained in every Borel $B$.

- $G/B$ is dimension 1, so there are infinitely many Borels in $G$

- So the above $\mathbb{G}_m$ is a singular torus of $GL(2, K)$ and the only one contained in $D(2, K)$
Defining $Z_\alpha$ and $G_\alpha$

- Let $\alpha : T \rightarrow \mathbb{G}_m$ for $\alpha \in \psi$
- Recall $T_\alpha = (\text{Ker } \alpha)^\circ$
- $T_\alpha$ is singular
- Set $Z_\alpha = C_G(T_\alpha)$
- By criterion 1, $Z_\alpha$ is not solvable
- Set $G_\alpha = Z_\alpha / R(Z_\alpha)$
- Since $Z_\alpha$ is not solvable, $G_\alpha$ is not the trivial group.
$G_\alpha$ is a semisimple group of rank 1

- Have seen $G_\alpha$ is not trivial
- It is a group modulo its radical
- So $G_\alpha$ is semisimple
- Note $T_\alpha \subseteq (Z_\alpha)^\circ \subseteq R(Z_\alpha)$
- So $T/T_\alpha$ is a maximal torus of $G_\alpha$
- Since $T_\alpha$ is a codimension one subtorus of $T$, we get that $G_\alpha$ has rank 1.