Integration (clarification)

Left estimate

Not necce that all rectangles have to be inside/outside

Formulas in general: Left est. for rectangles

$$\Delta x = \frac{b-a}{n}$$

$$L_n = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \ldots + f(x_{n-1}) \cdot \Delta x$$
Right estimate (5 rectangles)

Again not notice that all rectangles lie inside/outside.

In general, formula

\[ R_n = f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \]

\[ \Delta x = \frac{b-a}{n} \]
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} R_n \quad \text{(cont. func. } f \text{ on } [a, b])
\]

\[
= \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)
\]

**Example**

\[
\begin{align*}
&\text{graph of } f \\
&\int_0^y f(x) \, dx = \text{area under curve} \\
&= \frac{1}{2} x^2 y - \frac{\pi(2)^2}{4} \\
&= y - \pi
\end{align*}
\]
Section 4.3

Isaac Barrow: close connection between \( \int \) and \( \frac{d}{dx} \)
(Mentor: A. Newlin)

Fundamental Theorem of Calculus: explains this relation.

We define an "area" function

\[
g(x) = \text{area under } f \text{ for } [0, x] \\
= \int_0^x f(t) \, dt
\]

\[
g(0) = \int_0^0 f(t) \, dt = 0
\]

\[
g(1) = \int_0^1 f(t) = \frac{1}{2} \times 1 \times 2 = 1
\]

\[
g(2) = \int_0^2 f(t) \, dt - \int_0^1 f(t) \, dt = 1 + 1 \times 2 - 0.1 + 2 = 3
\]
\[ g(3) = g(2) + \frac{3}{2} \int f(t) \, dt \]

\[ \Rightarrow 3 + 1.3 = 4.3 \]

\[ g(4) = g(3) + \frac{4}{3} \int f(t) \, dt \]

\[ = 4.3 - 1.3 = 3 \]

\[ g(5) = g(4) + \frac{5}{4} \int f(t) \, dt \]

\[ = 3 - 1.3 = 1.7 \]
\[ g(x) = \int_{a}^{x} f(t) \, dt \]

= area under graph of \( f \) for \([a,x]\)

**Fundamental Theorem 1:** If \( f \) cont on \([a,b]\),
then \[ g(x) = \int_{a}^{x} f(t) \, dt \text{ as } x \leq b \]
is cont on \([a,b]\), and diff on \((a,b)\)
and \( g'(x) = f(x) \)
\[
g(x+h) - g(x) = \lim_{{h \to 0}} \frac{g(x+h) - g(x)}{h} = g'(x) = \lim_{{h \to 0}} \frac{A_h}{h}
\]
Find derivative \( g(x) = \int_0^x \sqrt{1+t^2} \, dt \)

\[ \text{By F.T.I} \quad g'(x) = \sqrt{1+x^2} \]

Find derivative \( h(x) = \int_1^x \sec t \, dt \)

Can't apply F.T.I directly

\[ u = x^4, \quad h(u) = \int_1^u \sec t \, dt \]

\[ h'(u) = \sec u \quad g(x) = h(u) = \int_1^x \sec t \, dt \]

\[ g'(x) = \sec u \cdot u' = (x^4) \cdot 4x^3 = (\sec x^4) \cdot 4x^3 \]
If $f$ cont. on $[a, b]$ then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where $F$ is any derivative of $f$.

\[ \text{[i.e. } F' = f \text{]} \]

---

**Proof**

$$g(x) = \int_{a}^{x} f(t) \, dt$$

$$g'(x) = f(x) \text{ by F.T part 1}$$

So $g$ is antiderivative of $f$.

And any antiderivative is $g + C$, an arbitrary constant.

$$\int_{a}^{b} f(x) \, dx = g(b) - g(a) = g(b) + C - (g(a) + C) = F(b) - F(a)$$
So find area under curve of $x^3$ for $[-2,1]$

$$\int_{-2}^{1} x^3 \, dx = \frac{x^4}{4} \bigg|_{-2}^{1} = \frac{1^4}{4} - \frac{(-2)^4}{4}$$

$$f(x) = x^3$$

$$F(x) = \frac{x^4}{4}, \text{ one antiderivative}$$
\[ F(x) \bigg|_a^b = F(b) - F(a) \quad \leftarrow \text{Notahun} \]

HW: Find \( \int_0^{\pi/2} \cos x \, dx \) using F.T

This makes it easier to find \( \int \) without estimating by rectangles.

Apply with caution

\[
\begin{align*}
\int_{-1}^{3} \frac{1}{x^2} \, dx &= \left[ \frac{-1}{x} \right]^3_{-1} \\
&= \frac{-1}{3} - \frac{-1}{-1} \\
&= -\frac{1}{3} - 1 \\
&= -\frac{4}{3}
\end{align*}
\]

Is this ok?

Think!

\[ \text{Area must be} \quad \text{true!} \]
So what is the flaw?

F.T only if f cont on \([a, b]\)

\[
\frac{1}{x^2} = \text{not cont on } [-1, 3]
\]

Fundamental thm of calculus

If \(f\) cont on \([a, b]\),

- \(g(x) = \int_{a}^{x} f(t) \, dt\), then \(g'(x) = f(x)\)
- \(\int_{a}^{b} f(x) \, dx = F(b) - F(a)\)

for \(F\) as anti-der. \(g, f\)

\([F' = f]\)
\[ \int_{\sin x}^{t} \sqrt{1 + t^2} \, dt = y \]

Find \( y' \)

Thus,

\[ y = \int_{0}^{\sin x} \sqrt{1 + t^2} \, dt - \int_{0}^{t} \sqrt{1 + t^2} \, dt \]

\[ = C - h(x) \]

(since #)

\[ h(x) = \int_{0}^{\sin x} \sqrt{1 + t^2} \, dt \]

\[ y' = 0 - h'(x) = -h'(x) = -\left(1 + \sin^2 x \right) \cos x \]

\[ u = \sin x \]

\[ h(x) = \int_{0}^{u} \sqrt{1 + t^2} \, dt \]

\[ h'(x) = \sqrt{1 + u^2} \cdot u' = \left(\sqrt{1 + \sin^2 x} \right) \cos x \]
HW - 13  (sec 4.3)  (sec 4.4)

Sec 4.3  3, 7, 10, 12, 13, 16, 17, 18
       21, 23, 25, 28, 31, 34, 49, 51
       54, 55, 56

Sec 4.4  19, 21, 23, 25, 27, 29, 35, 37

39

Be careful when you do this.
Split it up into 2 integrals.

55, 57.