Assume limits \( f, g \) exist.

**Laws**

**Sum of limits = limit of sum**

\[
\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \lim_{x \to a} f(x) + g(x)
\]

Caution: \( \lim_{x \to 0} x + \lim_{x \to 2} x \) are different.

**Similarity**

\[
\lim_{x \to a} (f - g)(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} f(x) \cdot g(x) = \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right)
\]

\[
\lim_{x \to a} c \cdot f(x) = c \lim_{x \to a} f(x)
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \quad \text{if} \quad \lim_{x \to a} g(x) \neq 0
\]

Examples:

\[
\begin{align*}
\lim_{x \to 2} (x - 2) &= 0 \\
\lim_{x \to 2} \frac{1}{x} &= \frac{1}{2} \\
(x - 2 + \frac{1}{x}) &= 0 + \frac{1}{2} = \frac{1}{2}
\end{align*}
\]
Constant limit

\[ \lim_{x \to a} c = c \]

Power

\[ \lim_{x \to a} (f(x))^n = \left( \lim_{x \to a} f(x) \right)^n \]

if \( \lim_{x \to a} f(x) \) exists

Ok, enough theory! Some hands on experience

\[ \lim_{x \to 5} 2x^2 - 3x + 4 \]

\[ = 2(5)^2 - 3(5) + 4 \]

\[ = 50 - 15 + 4 \]

\[ = 39 \]

No denominators so just plug in!

Actually, what are we doing

We find

\[ \lim_{x \to 5} 2x^2 = \lim_{x \to 5} 3x + \lim_{x \to 5} 4 \]

\[ = 2(5)^2 - 3(5) + 4 \]
Example 2 (with denominator)

\[ \lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{s - 3x} \]

Only problem point would be \( s - 3x = 0 \)

\[ x = \frac{5}{3} \]

but \( x \to -2 \), so we have no problem again

Just plug in!

\[ \text{Ans} = \frac{(-2)^3 + 2(-2)^2 - 1}{s - 3(-2)} = \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11} \]

Example 3 (with denominator)

\[ \lim_{x \to a} \frac{x - a}{x^2 - a^2} = \frac{1}{2a} \]

[Factorize, cancel problematic factors, plug in values]

Try \[ \lim_{x \to a} \frac{x - a}{x^3 - a^3} \]
I so what are we doing here

\[
   f(x) = \frac{x-a}{x^2-a^2}, \quad \text{if } x \neq a
\]

at a, we don't have \( f \) defined!

Define new function

\[
   g(x) = \frac{1}{x+a}, \quad \text{when } x \neq a
\]

And we see \( f(x) = g(x) \) everywhere except at \( a \).

so \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \)

\[
\]

\( m \to \)

\[
   \frac{\sqrt{1-x^2} - 1}{x}
\]

trick is to multiply

and divide by \( \sqrt{1-x^2} + 1 \)

\[
   \frac{\sqrt{1-x^2} - 1}{x} \cdot \frac{\sqrt{1-x^2} + 1}{\sqrt{1-x^2} + 1} \cdot \frac{\sqrt{1-x^2} + 1}{\sqrt{1-x^2} + 1}
\]

\[
   \frac{(1-x^2) - 1}{(1-x^2 + 1)} \cdot \frac{1}{x}
\]
\[
= \frac{-x^2}{x (\sqrt{1-x^2} + 1)}
\]

\[
= \frac{-x}{\sqrt{1-x^2} + 1}
\]

Plug in 0 now to get \( \lim_{t \to 0} \frac{0}{\sqrt{1} + 1} = \frac{0}{2} = 0 \)

\[\begin{array}{c}
\text{not ok, } \frac{0}{2}, \frac{0}{1.5} \text{ etc. ok } = 0 \\
\text{not ok, } \frac{2}{0}, \frac{1.5}{0} \text{ etc also ok! } \infty
\end{array}\]

What about \( \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \) \( \text{ NOT good!} \)

1. Plugging in 0 gives \( \times \)
2. What is the back? Multiply divide by \( \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} \)

\[
\frac{(\sqrt{t^2+9} - 3)}{t^2} \left( \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} \right) = \frac{t^2+9 - t^2}{t^2 (\sqrt{t^2+9} + 3)}
\]

\[= \frac{t^2}{t^2 (\sqrt{t^2+9} + 3)} \]
Now plug in $t=0$.

\[
\frac{1}{\sqrt{0+9+3}} = \frac{1}{6}
\]

\[
\lim_{x \to 0} |x| = 0
\]

What about

\[
\lim_{x \to 0} \frac{|x|}{x}
\]

Left limit $= -1$

Right limit $= 1$

So no limit!
greatest integer function

\[ \lceil x \rceil = \text{greatest integer less than or equal to } x \]

\[ \lceil 2.5 \rceil = 2 \]
\[ \lceil 2.3 \rceil = 2 \]
\[ \lceil 2.999 \rceil = 2 \]

Graph is cool!

\( \lim_{x \to 0^-} f(x) = -1 \) \( \lim_{x \to 0^+} f(x) = 0 \)

Limits exist at non-integer values, don't exist at integer values.

...
Here's a really weird function whose limits don't exist anywhere!

\[ \mathbb{Z} = \text{integers} \quad (\text{nice!}) \quad \{ -1, 2, 5, 7, \ldots \} \]
\[ \mathbb{Q} = \text{rationals}/ \quad \text{nice decimals} \]
\[ \frac{7}{5}, \quad \frac{1}{3}, \quad \frac{5}{2}, \quad \downarrow \]
\[ 0.3333\ldots, \quad 2.5 \]

You know the decimal expansion; it follows a pattern after some time.

\[ \mathbb{R} = \text{reals} \]

[Has some mysterious #s like \( \pi \), \( e \), \( \sqrt{2} \), etc. One day you will learn why \( \sqrt{2} \) is less complicated than \( \pi \)!]
Let us say

\[ f(x) = 0 \quad \text{if} \quad x \in \mathbb{Q} \]
\[ f(x) = 1 \quad \text{otherwise} \]

\[ \lim_{x \to 1} f(x) = ? \]

\[ f(1) = 0 \quad \text{as} \quad 1 \in \mathbb{Q} \]
but as \( x \to 1 \), you get 0's and 1's

no limit!

In fact, no left limit and no right limit.

Same for all points

Try to imagine the graph!

Key idea is that given any number, you can find a rational number very close to it, and also a nonrational number however close you want to get.
between any 2 #s, a \rightarrow b

can fit in both a rational (\in \mathbb{Q})

and an irrational number (\not\in \mathbb{Q})

So disillusion yourself of the idea that functions always look like

\[ \begin{array}{c}
\text{nice graph!}
\end{array} \]
useful ideas to find limits

"Sandwich theorem"

If \( f(x) \leq g(x) \leq h(x) \) for all \( x \) near \( a \) except maybe at \( a \)

and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then

\[ \lim_{x \to a} g(x) = L \]

we used this to show \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \).

Q: what is \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) \)?

\[ \left( \lim_{x \to 0} x^2 \right) \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \]

because \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) does not exist!
\[ x^2 \cdot (-1) \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2 \cdot 1 \]

\[ \uparrow \]

smaller
function

\[ \uparrow \]

larger function

However, \( \sin \left( \frac{1}{x} \right) \) is bounded between -1 and 1.

So

\[-x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2\]

\[ \lim_{{x \to 0}} x^2 \sin \left( \frac{1}{x} \right) = 0 \]

\[ \Rightarrow \lim_{{x \to 0}} x^2 \sin \left( \frac{1}{x} \right) = 0 \]
Subtle point

\[-1 \leq \sin \left( \frac{1}{x} \right) \leq 1 \]

\[
\int_{-\infty}^{0} \rightarrow x \to 0
\]

Can I just multiply by \( x \) to say

\[-x \leq x \sin \left( \frac{1}{x} \right) \leq x ?
\]

No!

if \( x = -\text{ve} \), then incorrect

2 cases

\( x \to 0^+ \)

\[-x \leq x \sin \left( \frac{1}{x} \right) \leq x
\]

\( x \to 0^- \)

Multiply by \( -\text{ve} \)

number reversing

inequalities

\[2 < 3 \]
\[x(-1) \]
\[-2 > -3 \]

so

\[(+x)(-1) \geq x \sin \left( \frac{1}{x} \right) \geq x \cdot x\cdot x
\]

so

\[-x \geq x \sin \left( \frac{1}{x} \right) \geq x
\]

\[\text{apply sandwich theorem}\]
Examples

\[
\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4} = \frac{1 + 4}{1 + 3 - 4} = \frac{5}{0} = \text{something undefined}
\]

1. Factorize:
\[
\frac{x(x - 4)}{(x - 4)(x + 1)}
\]

2. Is \(u \to +\infty\) or \(u \to -\infty\)

\[
\frac{x}{x + 1}
\]

as you plug in something close to \(-1\) and less than \(-1\).

Numerator \(\leq 0\)

Denominator \(= -2 + 1 = -1 < 0\)

So \(-ve / -ve = +ve\)