Continuity (sec 1.8)

We already defined continuity.

Cont at \( a \), graph doesn't break.

\[ \lim_{x \to a} f(x) \] exists and \( \lim_{x \to a} f(x) = f(a) \)

Discont at \( a \), graph breaks/jumps.

Either:

1. \( \lim_{x \to a} f(x) \) doesn't exist
2. \( \lim_{x \to a} f(x) \) exists but \( \lim_{x \to a} f(x) \) is not equal to \( f(a) \)
Which of the following functions are discontinuous?

a) \( x^2 - x - 2 \)

b) \( \frac{x^2 - x - 2}{x - 2} \)

c) \[ x \geq 3 \]

d) \( f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases} \)

e) \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases} \)

\( f(x) \) is nice at all points except maybe at 2.

Let us check what

\[ \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \]

polynomials are nice, continuous

\( f(2) \) is not even defined!

discontinuity

breaks at integers
discontinuity

\[ \begin{array}{c}
\text{discontinuity} \\
\text{breaks at 0} \\
\text{at 1} \\
\end{array} \]
\[
\frac{x^2 - x - 2}{(x-2)} = \frac{(x-2)(x+1)}{(x-2)} = x+1
\]

\[
\lim_{x \to 2} x+1 = 3
\]

But \( f(2) = 1 \)

So limit at 2 \( \neq f(2) \), so not cont

What about \( f(x) = 1 - \sqrt{1-x^2} \) on \([-1,1]\)?

\[
\begin{array}{c|c|c}
-1 & 1 & \text{meas} \\
\hline
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\]

So \( 1 - x^2 \geq 1 - 1 = 0 \)

So \( \sqrt{1-x^2} \) is ok for \([-1,1]\).
let us pick some $a$ in $(-1, 1)$

$$\lim_{{x \to a}} f(x) = \lim_{{x \to a}} 1 - \sqrt{1 - x^2}$$

$$= \lim_{{x \to a}} 1 - \lim_{{x \to a}} \sqrt{1 - x^2}$$

$$= 1 - \sqrt{\lim_{{x \to a}} (1 - x^2)}$$

$$= 1 - \sqrt{1 - a^2}$$

$$= f(a)$$

for end points $\to 1$

we can only see what $\lim_{{x \to -1^+}} f(x)$ and $\lim_{{x \to -1^-}} f(x)$ are

So check these are function values.
Some useful facts

If \( f, g \) are continuous at \( a \), so are

1) \( f + g \)
2) \( f \cdot g \)
3) \( c f \), \( c \) is a constant
4) \( f \cdot g \) if \( g(a) \neq 0 \)

Why?

\( f, g \) cont at \( a \), Thus means

\[
\lim_{x \to a} f(x) = f(a)
\]

\[
\lim_{x \to a} g(x) = g(a)
\]

So

\[
\lim_{x \to a} f(x) + g(x) = f(a) + g(a)
\]

using laws of limits
So this shows [Justification for most of our plug in points or limits]

Any polynomial is cont. everywhere.

ex: \[ ax^2 + 3(x) + 4 \]

\[ = 2(x)(x) + 3(x) + 4 \]

And \( f(x) = x \) = identity function is continuous.

And \( f(x) = c \) constant function is cont.

Builder blocks for polynomials = \{ f(x) = x \} and \( f(x) = c \)

D) Rational function is continuous in its domain.

\( \left( \frac{\text{polynomial}}{\text{polynomial}} \right) \)

So \( f(x) = \frac{x^2 - 2x - 3}{x - 2} \) is cont at everywhere except 2

At 2, we need to find limits and see what \( f(2) \) is
\[
\sin x
\]

\[\cot x\]

\[\csc x\]

Also \(\cos x\)!

\[\tan x\]

\[\cot x\]

HW!
So trigonometric functions are continuous in their domains!

Also root functions are continuous in their domains.

Where are these functions not continuous?

\[ \sqrt{x} + \frac{x+1}{x-1} + \frac{x+1}{x^2+1} = f(x) \]

Need to be defined, so find domain:

\[ \sqrt{x} \leftarrow [0, \infty) \]

\[ \frac{x+1}{x-1} \leftarrow \mathbb{R} \setminus \{1\} \]

\[ \frac{x+1}{x^2+1} \leftarrow \mathbb{R} \]

All denominators never vanish.

Common points: \( [0, \infty) \setminus \{1\} \)
and by our rule, the function is not in its domain.

<table>
<thead>
<tr>
<th>Note</th>
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<tbody>
<tr>
<td>$f(x) = \frac{1}{x-1}$</td>
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<tr>
<td>Domain of $f = \mathbb{R} \setminus {1}$</td>
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<th>$g(x) =$</th>
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<td>$\begin{cases} \frac{1}{x-1} &amp; x \neq 1 \ 2 &amp; x = 1 \end{cases}$</td>
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Another example:

$$\frac{\sin x}{2 + \cos x} = f(x)$$

can everywhere! as

domain = $\mathbb{R}$
denominator never vanishes as $2 + \cos x \neq 0$

This is a trigonometric function

-1 $\leq \cos x \leq 1$
If \( f \) is continuous at \( b \) and
\[
\lim_{x \to a} g(x) = b,
\]
then
\[
\lim_{x \to a} f(g(x)) = f(b).
\]

For \( x \) close to \( a \),
\[
\begin{align*}
g(x) & \text{ close to } b \\
f(g(x)) & \text{ close to } f(b)
\end{align*}
\]
basically
example:

\[ f(x) = e^x \quad \leftarrow \text{cont. everywhere so at } 4 \]

\[ g(x) = \text{some function} \quad \lim_{x \to 3} g(x) = 4 \]

what is

\[ \lim_{x \to 3} f(g(x)) = f\left( \lim_{x \to 3} g(x) \right) = f(4) = e^4 \]

here \( a = 3, \ b = 4 \)
If \( g \) is cont at \( 'a' \) and \( f \) is cont at \( 'g(a)' \), then \( f \circ g \) is cont at \( a \).

\[ x \to a \]

\[ g(x) \to g(a) \quad \text{[as \( g \) is cont at \( 'a' \)]} \]

\[ f(g(x)) \to f(g(a)) \quad \text{[as \( f \) is cont at \( g(a)' \)]} \]

So

\[ f \circ g \to f \circ g(a) \]

\( f \circ g \) is cont at \( a \).

(Ex) where \( \frac{1}{\sqrt{x^2 + 7} - 4} \) cont ?

Components:

\[ f(x) = \frac{1}{x} \quad g(x) = x - 4 \quad h(x) = \sqrt{x} \]

\[ k(x) = x^2 + 7 \]
So
\[ F(x) = \frac{1}{\sqrt{x^2+7} - 4} \]

= \( f \circ g \circ h \circ k(x) \)

Each \( f, g, h, k \) cont on \( \text{its domain} \)

So \( F \) cont on \( \text{its domain} \)

\[ F's \ domain \]
\[ \sqrt{x^2+7} - 4 \neq 0 \]
\[ \iff x^2 + 7 \neq 4^2 \]
\[ \iff \sqrt{x^2+7} \neq 4 \]
\[ \iff (\sqrt{x^2+7})^2 
eq 4^2 \]
\[ \iff x^2 + 7 \neq 16 \]
\[ \iff x^2 \neq 9 \]
\[ \iff x \neq \pm 3 \]

\[ \Rightarrow \ \text{IR} \ \setminus \{3, -3\} \ \text{or} \ (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \]