Final (version b)

N

14 Dec 2012

Name: ____________________________
Section: __________________________

This midterm totals 67 points (+ 3 extra credit points). You get 150 minutes to do it. Show all your work clearly, in the space provided. Good luck!

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<tr>
<th>Problem</th>
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1. (5 pts) A graph of $f$ is given below. Using that, for each of the following choose the right answer (A or B)

<table>
<thead>
<tr>
<th>$f'(2.5)$</th>
<th>(A) Positive</th>
<th>(B) Negative</th>
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<tr>
<td>$f(3) - f(2)$</td>
<td>(A) Positive</td>
<td>(B) Negative</td>
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<tr>
<td>$\int_{3}^{5} f(x)$</td>
<td>(A) Not 0</td>
<td>(B) Not 0</td>
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<tr>
<td>$\int_{-2}^{-3} f(x)dx$</td>
<td>(A) Positive</td>
<td>(B) Negative</td>
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If $g(x) = \int_{2}^{x} f(t)dt$
then $g'(2.5)$ is (A) Positive (B) Negative
2. Find the following integrals

(a) (5 pts) \( \int_0^{\frac{\pi}{2}} (-\sin x)^7 (\cos x) \, dx \)

\[
\begin{align*}
  u &= -\sin x \\
  du &= \cos x
\end{align*}
\]

\[\int_0^{\frac{\pi}{2}} -u^7 \, du = \left[ -\frac{u^8}{8} \right]_0^1 = -\frac{1}{8} \]
(b) (3 pts) \( \int x^4(\sqrt{x^5 + 3}) \, dx \)

\[ u = x^5 + 3 \]
\[ du = (5x^4) \, dx \]

\[ \int \sqrt{u} \, \frac{du}{5} = \int \frac{u^{\frac{1}{2}}}{5} \, du \]

\[ = \frac{1}{5} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \]
\[ = \frac{2}{15} u^{\frac{3}{2}} + C \]
\[ = \frac{2}{15} (x^5 + 3)^{\frac{3}{2}} + C \]

(c) (2 pts) \( \int_1^{10} (\sin^2 \theta + \cos^2 \theta) \, d\theta \)

\[ = \int_1^{10} 1 \, d\theta \]
\[ = \theta \bigg|_1^{10} \]
\[ = 10 - 1 \]
\[ = 9 \]
3. (5 pts) Find a point \((x, y)\) lying on \(4x^2 + y^2 = 4\) that is farthest away from the point \((1, 0)\).

[Hint: The square of the distance between points \((x_1, y_1)\) and \((x_2, y_2)\) is given by the formula \(d = (x_1 - x_2)^2 + (y_1 - y_2)^2\)]

\[
d(x) = (x - 1)^2 + y^2
\]
\[
= x^2 + 1 - 2x + (y - y_2)^2
\]
\[
= -3x^2 - 2x + 5
\]
\[
d'(x) = -6x - 2 \quad \Rightarrow \quad 0
\]
\[
-6x = 2 \quad \Rightarrow \quad x = -\frac{1}{3}
\]

\[
y = \sqrt{4 - 4 \left(\frac{1}{3}\right)}
\]
\[
= \sqrt{\frac{32}{9}}
\]
\[
= \pm \frac{4\sqrt{2}}{3}
\]

Extremals:

\[
d'(x) = -6(x + \frac{1}{3})
\]

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<td></td>
<td>&gt;0</td>
<td>(\frac{1}{3})</td>
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\(a \approx \text{max!}\)

\[A_{10} = \left(-\frac{1}{3}, \pm \frac{4\sqrt{2}}{3}\right)\]
4. (5 pts) Find the equation of the tangent to $x^2 + xy + y^2 = 7$ at $(1, 2)$

\[ 2x + xy' + y + 2yy' = 0 \]

\[ y'(x + 2y) = -2x - y \]

\[ y' = \frac{-2x - y}{x + 2y} \]

\[ y' \bigg|_{(1, 2)} = \frac{-2 - 2}{1 + 5} = \frac{-4}{6} = m \]

\[ y = mx + c \]
\[ y = \frac{-4}{6}x + c \]
\[ 2 = \frac{-4}{6} + c \]
\[ c = 2 + \frac{4}{6} = \frac{14}{6} \]

\[ y = \frac{-4}{6}x + \frac{14}{6} \]

5. (5 pts) Find the following limits

\( \text{Answ.} \quad y = \frac{-4}{6}x + \frac{14}{6} \)

(a) \( \lim_{x \to 6} \frac{x^2 - 2x - 24}{x^2 - 7x + 6} \)

\[ = \lim_{x \to 6} \frac{(x - 6)(x + 4)}{(x - 6)(x - 1)} \]

\[ = \frac{10}{5} = 2 \]

(b) \( \lim_{x \to 0} \frac{\sin 3x}{x} \)

\[ = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3 \]

\[ = 1 \times 3 = 3 \]
6. This is a three-part question.

(a) (3 pts) Draw a rough sketch of the curves \( y = 2x - x^2 \) and \( y = x^3 \) in the first quadrant. Find the intersection points and mark them clearly on the figure. Which curve lies on the top in the first quadrant?

(b) (3 pts) Find the area of the region \( R \) bounded by \( y = x^3 \), \( y = 2x - x^2 \), \( x = 0 \) and \( x = 1 \).

\[
\begin{align*}
\int_0^1 2x - x^2 - x^3 &= \int_0^1 2x - x^2 - x^3 \\
&= \int_0^1 2x - x^2 - x^3 \\
&= \left[ \frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \\
&= \frac{1}{3} - \frac{1}{4} \\
&= \frac{12 - 4 - 3}{12} \\
&= \frac{5}{12}
\end{align*}
\]
(c) (4 pts) Imagine that you rotate the region \( R \) in the previous question around the \( x \)-axis. Find the volume of the resulting solid.

[Hint: The vertical cross-sections are rings]

\[
A(x) = \pi \left( (2x - x^2)^2 \right) - \pi (x^3)^2
\]

\[
= \pi \left( 4x^2 + x^4 - 4x^3 \right) - \pi (x^6)
\]

\[
= \pi \left( 4x^2 + x^4 - 4x^3 - x^6 \right)
\]

\[
\int_0^1 A(x) \, dx = \pi \left( \frac{4x^2}{3} + \frac{x^5}{5} - \frac{4x^4}{4} - \frac{x^7}{7} \right) \bigg|_0^1
\]

\[
= \pi \left( \frac{4}{3} + \frac{1}{5} - 1 - \frac{1}{7} \right)
\]

\[
\approx \frac{8}{\pi} \left( \frac{140 + 21 - 105 - 15}{105} \right)
\]

\[
\approx \frac{105}{105} (41)
\]
7. (5 pts) Two men start walking from the same point at 12:00 noon. One travels south at 6 mi/hr and the other travels west at 8 mi/hr.

(a) What is the distance between them at 10:00 P.M

(b) At what rate is the distance between them increasing at 10:00 P.M

\[
\begin{align*}
\frac{dz}{dt} & = \frac{2(60)(8) + 2(80)(6)}{200} \\
& = \frac{8^2 + 6^2}{10} \\
& = 10 \\
A_{w} & = 10 \text{ mi/hr}
\end{align*}
\]
8. (a) (2 pts) State the fundamental theorem of calculus (both the parts)

\[ f \text{ is cont on } [a, b] \]
\[ g(x) = \int_a^x f(t) \, dt \]

1. Then \( g'(x) = f(x) \)

2. \[ \int_a^b f(t) \, dt = F(b) - F(a) \] for any antiderivative \( F \).

(b) (1 pt) If \( g(x) = \int_0^x (\tan t) \, dt \), what is \( g' \left( \frac{\pi}{4} \right) \)?

\[ g'(x) = \tan x \]
\[ g' \left( \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1 \]

(c) (2 pts) If \( h(x) = \int_0^{x^3} (\tan t) \, dt \), what is \( h'(x) \)?

\[ u = x^3 \]
\[ h'(x) = \tan u \cdot u' \]
\[ = (\tan x^3) \cdot (3x^2) \].
9. A ant started moving from the origin in a straight line at $t = 0$ seconds. Its initial velocity was 2 mm/s and it stopped moving at $t = 3$ seconds. Its acceleration was found to be the function $a(t) = 2t - 3$

(a) (3 pts) Find its velocity and displacement functions

$$v(t) = \frac{2t^2}{2} - 3t + 2 \quad (v(0) = 2)$$

$$= t^2 - 3t + 2$$

$$s(t) = \frac{t^3}{3} - \frac{3t^2}{2} + 2t \quad (s(0) = 0)$$

(b) (3 pts) Find the absolute minimum and maximum displacements for this ant

$$s'(t) = v(t) = (t - 2)(t - 1)$$

Critical $t$: 1, 2

End $t$: 0, 3

$$t = \begin{array}{c|c|c|c}
0 & 1 & 2 & 3 \\
\hline
s(t) = & 0 & \frac{5}{6} & \frac{2}{3} & \frac{1}{2} \\
\end{array}$$

Also, $s(\infty) = \infty$

(c) (1 pt) If a friendly praying mantis is waiting at the point (1,0) [i.e. +1 mm to the right of the origin], will the ant run into it? Explain.

Yes. (Wrong!)
10. (10 pts) Say whether the following statements are true or false. **Give proper reasons to justify your answer**

(a) **True or False**: $f(x) = \sin x$ is an decreasing function on the interval $(0, \pi)$

![Graph showing increasing function in $(0, \pi/2)$]

$f'(x) = \cos x > 0$ on $(0, \pi/2)$

so $f$ is increasing on $(0, \pi/2)$

(b) **True or False**: $\int_0^{\pi/2} \tan x \, dx \geq \frac{\pi}{4}$

![Graph showing integral with upper and lower bounds]

$0 \leq \tan x \leq 1$

on $[0, \pi/4]$

so $\int_0^{\pi/4} \tan x \, dx \leq \frac{\pi}{4}$

False
(c) **True or False**: \( f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 10 \) has a local maxima at \( x = 1 \)

\[
\begin{align*}
\text{True} \\
\frac{df}{dx} &= \frac{3x^2}{3} - \frac{6x}{2} + 2 = x^2 - 3x + 2 \\
&= (x - 2)(x - 1) \\
\frac{d^2f}{dx^2} &= 2x - 3 \\
\frac{d^3f}{dx^3}(1) &= 2(1) - 3 = -1 < 0, \text{ so local max at } x = 1
\end{align*}
\]

(d) **True or False**: \( f(x) = x^{1000} \) is concave down on the interval \((0,\infty)\)

\[
\begin{align*}
\text{False} \\
\frac{df}{dx} &= 1000x^{999} \\
\frac{d^2f}{dx^2} &= 999000x^{998} > 0 \text{ on } (0,\infty) \\
\text{So concave up}
\end{align*}
\]

(e) **True or False**: \( \int_{\frac{\pi}{2}}^{\pi} (\cos x)dx = \int_{\frac{\pi}{2}}^{0} (\cos x)dx \)

\[
\begin{align*}
\text{LHS} &= \int_{0}^{\pi/2} \cos x \, dx = -\sin x \bigg|_{0}^{\pi/2} = -\sin \frac{\pi}{2} - (-\sin 0) \\
&= -1 \\
\text{RHS} &= - \text{LHS} = 1 \\
\text{False!}
\end{align*}
\]
11. Extra credit (3 pts) Find \( \int_{-3}^{0} \sqrt{9 - x^2} \, dx \).

\[
\text{Area} = \frac{11}{4} \left( \frac{3}{2} \right)^2 = \frac{9}{4} \pi
\]

\[
\left( \frac{1}{4} \sin \theta \right) \, d\theta
\]