Midterm I  (Version a)

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28 Sep 2012

Name : 
Section : 001 / 101

This midterm totals 40 points. You get 50 minutes to do it. Show all your work clearly, in the space provided. Good luck!
1. (3 pts) If \( f(x) = \sin x \) and \( g(x) = \frac{1}{x} \), fill up the blanks.

(a) Domain of \( f(x) \) is \( \mathbb{R} \),

(b) Domain of \( g(x) \) is \( \text{everywhere but } 0 \).

(c) Find the function \( g \circ f : \frac{1}{\sin x} \)

(d) Domain of \( g \circ f \) is \( \text{everywhere but multiples of } \pi \).

2. (3 pts) Look at the following figure and answer the following questions.

(a) \( \lim_{x \to 3^-} f(x) = 3 \)

(b) \( \lim_{x \to 3^+} f(x) = 2 \)

(c) \( f(3) = 2 \)
3. Find the following limits of the following. Write down all your steps clearly

(a) (5 pts) \( \lim_{t \to 1} \left( \frac{2t-2}{3t^2+2t-5} \right) \)

\[
\frac{2t-2}{3t^2+2t-5} = \frac{2(t-1)}{(3t+5)(t-1)} = \frac{2}{3t+5}
\]

\[
\sqrt{3t^2+2t-5} = 3t^2 + 5t - 3t - 5 = t(3t+5) - 1(3t+5) = (t-1)(3t+5)
\]

\[
\therefore \lim_{t \to 1} \frac{2}{3t+5} = \frac{2}{(3\times1)+5} = \frac{2}{8} = \frac{1}{4}
\]

(b) (2pts) \( \lim_{x \to 0} \frac{\sin x}{7x} \)

\[
\lim_{x \to 0} \frac{\sin x}{7x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \right) \times \frac{1}{7}
\]

\[
= \frac{1}{7} \lim_{x \to 0} \frac{\sin x}{x}
\]

\[
= \frac{1}{7} \times 1 = \frac{1}{7}
\]
4. (2 pts) Find the value of $a$ so that the following function is continuous. Write down your final answer in the blank and also show clear steps.

$$f(x) = \begin{cases} 
-a\sqrt{x} & \text{if } 4 \leq x \\
2 & \text{if } x < 4 
\end{cases}$$

Answer: $a = -1$

Left limit at 4: \( \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} -a\sqrt{x} = -a\sqrt{4} = -2a \)

Right limit at 4: \( \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} 2 = 2 \)

5. (3 pts) Show that the equation \( x^3 = x + 1 \) has a solution in between -2 and 2. Do not forget to mention which result you use to prove this!

\[ f(x) = x^3 - x - 1 \]

\[ f(-2) = (-2)^3 - (-2) - 1 = -8 + 2 - 1 = -7 < 0 \]

\[ f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5 > 0 \]

By IVT, there is a root.
6. (10 pts) Find the derivative of $f(x) = \sqrt{x - 1}$ at $x = 5$ using the difference quotient method. And hence, find the equation of the tangent to $f(x)$ at $(5, 2)$. Write down your final answers in the blanks given below (otherwise you will be penalized!) and also show all the steps clearly.

$\frac{f'(5)}{1}$

- Equation of the tangent at $(4, 2)$ is $y = \frac{1}{4}x + \frac{3}{4}$.

\[\frac{f'(5)}{1} = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \to 0} \frac{\sqrt{5 + h - 1} - \sqrt{5 - 1}}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h} = \lim_{h \to 0} \frac{(\sqrt{4 + h} - 2)^2}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{2 + 2} = \frac{1}{4}
\]

Slope of tangent $= \frac{1}{4} = m$.

$y = mx + c$

$y = \frac{1}{4}x + c$

$x = 5$, $y = 2$

$2 = \frac{1}{4}(5) + c$, $c = 2 - \frac{5}{4} = \frac{8}{4} - \frac{5}{4} = \frac{3}{4}$

$y = \frac{1}{4}x + \frac{3}{4}$
7. (a) (2 pts) Write down the multiplication and division rules of differentiation in the space given below.

i. \((fg)' = f'g + fg'\)

ii. \(\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\)

(b) (5 pts) Find derivative of \(h(x) = \sqrt{x}(x^2 + 3\sin x)\) and write down your steps clearly.

\[
\begin{align*}
    f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\
    f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \\
    g(x) &= x^2 + 3\sin x \\
    g'(x) &= 2x + 3\cos x
\end{align*}
\]

\[
(fg)' = f'g + fg' = \frac{1}{2\sqrt{x}} (x^2 + 3\sin x) + \sqrt{x} (2x + 3\cos x)
\]

\[
= \frac{2x^2 + 3\sin x + 2x (2x + 3\cos x)}{2\sqrt{x}}
\]

\[
= \frac{5x^2 + 3\sin x + 6x \cos x}{2\sqrt{x}}
\]
(c) (5 pts) Find the derivative of $h(x) = \frac{\cos x}{x+1}$ and write down your steps clearly.

\[
\begin{align*}
    f(x) &= \cos x \\
    g(x) &= x+1
\end{align*}
\]

\[
\frac{f'(x)}{g(x)} = -\sin x
\]

\[
\begin{align*}
    \left(\frac{f}{g}\right)' = & \quad \frac{f'g - fg'}{g^2} \\
    = & \quad \frac{(-\sin x)(x+1) - (\cos x)(1)}{(x+1)^2} \\
    = & \quad \frac{-x\sin x - \sin x - \cos x}{(x+1)^2}
\end{align*}
\]