Midterm II

9 Nov 2012

Name: __________________________________________
Section: _________________________________________

This midterm totals 30 points. You get 50 minutes to do it. Show all your
work clearly; in the space provided. Good luck!

1. (3 pts) Find the derivative of \( f(x) = \frac{x-1}{x-2} \). Is \( f \) increasing or decreasing
on the interval \((3, \infty)\)?

\[
\frac{d}{dx} f(x) = \frac{(x-1)'(x-2) - (x-2)'(x-1)}{(x-2)^2} \\
= \frac{1(x-2) - 1(x-1)}{(x-2)^2} \\
= \frac{x-2 - x + 1}{(x-2)^2} \\
= \frac{-1}{(x-2)^2} < 0 \text{ on } (3, \infty)
\]

So \( f \downarrow \)
2. You are given the equation of a curve $\sin y = x^2 - x$

(a) (2 pts) Find $y'$

$$\cos y \quad y' = 2x - 1$$

$$y' = \frac{2x-1}{\cos y}$$

(b) (2 pts) Using this, find the equation of the tangent to the above curve at $(0,0)$

$$x=0, \quad y=0$$

$$y' = \frac{-1}{\cos 0} = \frac{-1}{1} = -1 = m$$

$y = mx+c$

$y = -x + c$

$0 = 0 + c \quad \therefore c = 0$

$$y = -x$$

(c) (1 pt) For what value of $x$ is the tangent at $(x, f(x))$ horizontal?

$$m = \frac{2x-1}{\cos y}$$

If $x = \frac{1}{2}$, $m = 0 = \text{Slope}$
3. You are given a function $f$ such that $f(0) = 0$ and $f'(x) = x^2 - 3x + 2$.

(a) (2 pts) What are the critical points for $f$?

$$f'(x) = x^2 - 3x + 2$$
$$= (x-2)(x-1)$$

critical pts : 2, 1

(b) (3 pts) Find all the local maxima and minima

$$f''(x) = 2x - 3$$

$f''(2) = 2(2) - 3 = 1 > 0$ local min

$f''(1) = 2(1) - 3 = 2 - 3 = -1 < 0$ local max

Second derivative test !

(c) (2 pts) Find $f$

$$f = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$$

$f(0) = C = 0$ : $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x$
4. (5 pts) The base of a right triangle is decreasing at the rate of 1 cm/s while the height of the right triangle is found to be increasing at the rate of 2 cm/s. When the base is 3 cm and the height is 4 cm, find out the length of the hypotenuse and the rate at which it is changing.

\[
\frac{db}{dt} = -1 \\
\frac{da}{dt} = 2
\]

\[a^2 + b^2 = c^2\]

\[4^2 + 3^2 = c^2 \implies 16 + 9 = 25 = c^2 \implies c = 5\]

\[2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}\]

\[2(4)(2) + 2(3)(-1) = 2(5) \frac{dc}{dt}\]

\[\frac{dc}{dt} = \frac{16 - 6}{10} = \frac{10}{10} = 1 \text{ cm/s}\]

The hypotenuse increases at a rate of 1 cm/s.
5. (5 pts) Find two positive integers in the interval \([0, 1600]\) such that the sum of the first number and four times the second number is 1600 and the product of the numbers is as large as possible.

\[
\begin{align*}
x + 4y &= 1600 \\
x &= 1600 - 4y
\end{align*}
\]

\[
f(y) = xy = (1600 - 4y)y = 1600y - 4y^2
\]

\[
f'(y) = 1600 - 8y
\]

Critical point: \(y = \frac{1600}{8} = 200\)

\[
y = 0 \\
200
\]

\[
x = 800, \ y = 800
\]

Absolute max,
6. (a) (1 pt) What is the slope of the line joining \((a, f(a))\) and \((b, f(b))\)

\[
m = \frac{f(b) - f(a)}{b - a}
\]

(b) (2 pts) If \(f(0) = 0\) and \(f'(x) = 3\) for every \(x < 4\). Predict the value of \(f(2)\) using MVT.

By MVT

\[
\frac{f(2) - f(0)}{2 - 0} = f'(c) \quad \text{for some } c \in (0, 2)
\]

\[
\Rightarrow \quad \frac{f(2) - 0}{2} = \frac{f(2)}{2} = f'(c)
\]

\[
\Rightarrow \quad f(2) = 2f'(c)
\]

\[
= 2 \times 3
\]

\[
= 6
\]

\(-f(2) = 6\)
7. (2 pts) Find the following limit

\[
\lim_{x \to \infty} \frac{3x^2 + 2x + 5}{2x^2 + 5x + 7}
\]

\[
\lim_{x \to \infty} \frac{3x^2 + 2x + 5}{2x^2 + 5x + 7} = \lim_{x \to \infty} \frac{3 + \frac{2}{x} + \frac{5}{x^2}}{2 + \frac{5}{x} + \frac{7}{x^2}}
\]

\[
= \frac{3}{2}
\]

divide num, denom by \(x^2\)