then what does \( f(x) \) approach as \( x \) gets closer to 3?

\[
\lim_{x \to 3} f(x) = 2
\]
[we have found limit of a constant function at a point]

\[
\lim_{\substack{x \to 1 \\ x \neq 3}} f(x) = 2
\]

1. \( \lim_{x \to 1} \frac{x-1}{x^2-1} = ? \)

   **Step 0:** Try to plug in 1, we get \( \frac{0}{0} \) BAD!

   **Step 1:**
   \[
   \frac{x-1}{x^2-1} = \frac{1}{x+1}
   \]

   **Step 2:**
   \[
   \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2} \quad \leftarrow \text{plugging in 1 is OK}
   \]

__NOW__
Cautiion!

\[ g(x) = \frac{x-1}{x^2-1} \quad \text{if} \quad x \neq 1 \]

\[ g(1) = 3 \]

What is \( \lim_{x \to 1} g(x) ? \)

Is it still \( \frac{1}{2} \) or is it 3?

It is still \( \frac{1}{2} \)

"Limit value doesn't depend on function value at that point."
\[ \lim_{x \to 0} \frac{\sin x}{x} \]

1. Plugging in 0 gives \( \frac{0}{0} = \text{BAD!} \)

2. Circle of radius 1

\[
\sin \theta = \frac{AC}{AB} = \frac{AC}{1}, \quad \text{so} \quad AC = \sin \theta
\]

\[
\cos \theta = \frac{AC}{AD} = \frac{\sin \theta}{AD}, \quad \text{so} \quad AD = \frac{\sin \theta}{\cos \theta} = \tan \theta
\]

\[
\sin \theta \leq \theta \leq \tan \theta
\]

\[
\frac{\sin \theta}{\sin \theta} \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}, \quad 1 \leq \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}
\]

\[
\lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1
\]
Find
\[ \lim_{x \to \pi} \frac{\sin x}{\pi - x} \]

\[ \sin (\pi - x) = \sin x \]

\[ \lim_{x \to \pi} \frac{\sin (\pi - x)}{\pi - x} = \lim_{\pi - x \to 0} \frac{\sin (\pi - x)}{\pi - x} = 1 \]

(3) what about \[ \lim_{x \to 0} \frac{\pi}{x} = \infty \]

\[ \lim_{x \to 0} \sin \left( \frac{\pi}{x} \right) \]

Guesses?

\[ x \to 0 \quad \pi / x \to \infty \]

so \( \sin \left( \frac{\pi}{x} \right) \) rotates through all values \([-1, 1]\)

\[ \sin \left( \frac{\pi}{x} \right) \]

\[ \text{So limit doesn't always exist} \]
One sided limits

\[ f \]

\((0,5)\)  
\((0,4)\)  
\((0,2)\)

\((3,0)\)

Reading graph

\[ f(x) = 2 \quad \text{if} \quad x < 3 \quad \Rightarrow \quad f(2.9999) = 2 \]

\[ f(3) = 4 \]

\[ f(x) = 5 \quad \text{if} \quad x > 3 \quad \Rightarrow \quad f(3.00001) = 5 \]

\(\ast\) denotes \( f \) is not that value at the point

Left limit
\[ \lim_{{x \to 3^-}} f(x) = 2 \]

Right limit
\[ \lim_{{x \to 3^+}} f(x) = 5 \]
So left limit ≠ right limit

So \( \lim_{x \to 3} f(x) \) does not exist.

\[ x \to 3 \]

Note

If left limit = right limit, then we say limit exists and is equal to left, right limits

\[ \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) \]

If left limit ≠ right limit

then limit does not exist

None of them may be \( = f(a) \)

Limit \( f(x) \) and \( f(a) \) are different. Sometimes they are equal, sometimes not.
\[ \lim_{x \to 0^+} f(x) = 1 \]

\[ \lim_{x \to 0^-} f(x) = 1 \]

so \( \lim_{x \to 0} f(x) = 1 \)

So

* If right limit = left limit, then limit exists

* Neither of them need to be \( f(\text{point}) \)

caution!
Discontinuous at 'a'

Royally: graph of f breaks at x = a

Pictorially:

Practical test: You can't draw graph without lifting your hand from the paper

Mathematically:

Either

1. \( \lim_{x \to a} f(x) \) does not exist

or

2. \( \lim_{x \to a} f(x) \neq f(a) \)

You can draw the graph without lifting hand from paper

\[ \lim_{x \to a^-} f(x) = f(a) \]

\[ \lim_{x \to a^+} f(x) = f(a) \]

so in particular limit exist.
\[ \tan \frac{\pi}{2} = \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \infty ! \quad \text{Left limit} \]

\[ \tan \text{ (second quadrant) } -\text{ve} ! \]

\[ \text{so } -\infty \quad (\text{as seen from graph}) \]

\[ \lim_{\theta \to \frac{\pi}{2}^-} \tan \theta = \infty \]

\[ \lim_{\theta \to \frac{\pi}{2}^+} \tan \theta = -\infty \]
\( \varphi: \text{ Draw graph of } e^x \)

\[
\begin{align*}
\lim_{x \to 0^+} e^x &= 1 \\
\lim_{x \to 0^-} e^x &= 1 \\
\lim_{x \to -\infty} e^x &= 0 \\
\lim_{x \to \infty} e^x &= \infty
\end{align*}
\]

\( \text{and also } e^0 = 1 \) so \( e^x \) continuous at 0.

\[
\varphi: \lim_{x \to 0.2} \frac{1}{x^2} = \frac{1}{0.04} = 4
\]

\[
\lim_{x \to 0} \frac{1}{x^2} = \infty
\]

\[
\varphi: \lim_{x \to 0^+} \frac{1}{x} = \infty \\
\lim_{x \to 0^-} \frac{1}{x} = -\infty
\]
Vertical asymptotes

Vertical line: $x = a$ is asymptotic at $a$ if

1. $\lim_{x \to a^-} f(x) = \infty$
2. $\lim_{x \to a^+} f(x) = \infty$
3. $\lim_{x \to a^-} f(x) = -\infty$
4. $\lim_{x \to a^+} f(x) = -\infty$
Q: what is an asymptote of \( \tan \) ?
   what is \( 'a' \)? what other \( 'a' \) values can you think of?

Q: Is there a vertical asymptote for \( \sin x \) or \( \cos x \)?

Q: what about \( \cot x \)?

Graph \( \cot x \)

\[ \pi - \text{periodic} \]
Questions

\( 0 \lim_{x \to 3^+} \frac{2x}{x-3} \quad \text{Value to plug in} \quad 4, 3.5 \quad \frac{8}{1} \quad \frac{7}{0.5} = 14 \ldots \)

\( 1 \lim_{x \to 3^-} \frac{2x}{x-3} \quad 2, 2.5 \quad -4 \quad \frac{-5}{0.5} = -10, \ldots \)

Guess: \( \lim_{x \to 3^+} \frac{2x}{x-3} = \infty \)

\( \lim_{x \to 3^-} \frac{2x}{x-3} = -\infty \)

\( \frac{2x}{x-3} \) numerator: lies around 6

denominator: gets very close to 0

So \( \frac{2x}{x-3} \) becomes very large \(+ve\) if \( x > 3 \)

becomes very \(-ve\) if \( x < 3 \)
Find vertical asymptotes of the following function:

\( f(x) = \frac{2}{x^2 - 1} \)

1. Right/left when does \( \lim_{x \to ?} f(x) = \pm \infty \)
2. When denominator vanishes: \( x^2 - 1 = 0 \) \( x = \pm 1 \)

\( \lim_{x \to 1^+} f(x) = \infty \)
\( \lim_{x \to 1^-} f(x) = -\infty \)
\( \lim_{x \to -1^+} f(x) = -\infty \)
\( \lim_{x \to -1^-} f(x) = +\infty \)

Asymptotes are \( x = -1 \)
\( x = 1 \)
Finding when limit is \( \infty \) or \(-\infty\)

\[
\begin{align*}
\text{limit} & \quad \frac{x+2}{x+3} = \infty \\
x \to -3^- & \quad \frac{-3+2}{-3+3} = \frac{-1}{0}
\end{align*}
\]

1. Plug in \( x = -3 \), we get \( \frac{-3+2}{-3+3} = \frac{-1}{0} \)

So we guess the limit must be \( \infty \) or \(-\infty\)

I as \( x \) gets closer to \(-3\), \( x+2 \) gets closer to \(-1\) and denominator gets closer to \(0\)

\[
\frac{-1}{0} \quad \text{blows up}
\]

2. How to determine \( \text{sign} \)?

\[
\begin{align*}
x & \quad \xrightarrow[]{-3} \\
-5 & \quad -4 \quad -3.5 \quad -3
\end{align*}
\]

\[
\begin{align*}
x = -4 & , \quad \frac{x+2}{x+3} = \frac{-4+2}{-4+3} = \frac{-2}{-1} = 2 \\
x = -3.5 & , \quad \frac{x+2}{x+3} = \frac{-3.5+2}{-3.5+3} = \frac{-1.5}{-0.5} = 3
\end{align*}
\]
\[ x < -3 \quad \text{so} \quad x < -2 \]

then \( x + 3 < 0 \) and \( x + 2 < 0 \)

\[
\frac{\text{number} < 0}{\text{number} < 0} = \frac{\text{number} > 0}{\text{number} > 0}
\]

\[ \text{so} \quad +\infty \]

(2) \[
\lim_{x \to 0} \frac{x-1}{x^2(x+2)} = (-\infty)
\]

1. Plugging in \( x = 0 \), you get \( \frac{-1}{0} \)

So it is \(+\infty\) or \(-\infty\)

2. What is the sign?

As \( x \) is very close to 0, what is \( x-1 \)

close to? It is close to \(-1\), so numerator sign is \(-\text{ve}\)

\[
\begin{array}{c}
\frac{\text{-ve}}{\text{+ve}} \\
= \text{-ve}
\end{array}
\]

as \( x \to 0 \) denominator sign = ?

\( x^2 \) is always \( \text{+ve} \)

\( x+2 \) as \( x \to 0 \) is \( +2 \), so the denominator is \( +\text{ve} \)
\[
\lim_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4} = -\infty
\]

0. Plugging in 2 gives \(\frac{0}{0}\)

1. Factorize!

\[
\frac{x^2 - 2x}{x^2 - 4x + 4} = \frac{x(x-2)}{(x-2)^2}
\]

\[
= \frac{x}{x-2}
\]

2. Plugging in \(x = 2\) gives \(\frac{2}{0}\)

so limit is \(\infty\) or \(-\infty\)

3. What is the sign?

\[
x \to 2^-
\]

Say \(x = 1\),

\[
\frac{x}{x-2} = \frac{1}{1-2} = -1
\]

If \(x < 2\), and close to 2, numerator sign = \(+ve\)

denominator sign = \(-ve\)

\(+ve/-ve = -ve\)
Physics!

**Special Relativity:**Nothing travels faster than light mass & energy

mass of a particle with velocity \( v \)

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

as \( v \to c^- \), what happens to \( m \)?

\[
\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \to \infty
\]