Calculus (pebble/stone used for counting) Latin!

Credits: Leibniz, Newton (17th century) + Greeks! China! India! Middle East!

Popular functions!

\[ f(x) = e^x \]

\[ f(x) = \sin(x) \]

TRICK ALERT: Not a function!
Exponentials and Logarithms

Exponentiation = repeated multiplication

\[ 2^5 = \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ times}} \]

So \[ a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}} \], \( n = \text{natural number} \)

\[ a^0 = ? \], by definition it is 1

\[ 2^{-5} = \frac{1}{2^5} \]

So \[ a^{-n} = \frac{1}{a^n} \]

What about raising to fractions?

\[ a^{\frac{1}{5}} = \sqrt[5]{2} = 5^{\text{th}} \text{ root of 2} \]

\[ a^{\frac{1}{n}} = \sqrt[n]{a} = n^{\text{th}} \text{ root of } a \]

\[ 2^{\frac{3}{5}} = \left(2 \times 2 \times 2\right)^{\frac{1}{5}} = \sqrt[5]{8} \]

\[ = (2^{\frac{1}{5}})^3 = 2^{\frac{1}{5}} \times 2^{\frac{1}{5}} \times 2^{\frac{1}{5}} \]
\[ \frac{p}{a^2} = a^{\frac{p}{2}} = \sqrt[p]{a} \cdot \sqrt[p]{a} \ldots \sqrt[p]{a} \]

p times

if you are approximating roots, answers might differ slightly.

Rational numbers account for a lot!

\[ \leftarrow \quad \mathbb{R} \quad \rightarrow \]

\[ \leftarrow \quad \mathbb{Q} = \text{rationals} \quad \rightarrow \]

Challenge question: \( \overline{234234234234} \ldots \) is also a rational number!

Find it.

Easier example \( 0.9999 \ldots = 1 \)

Non rational numbers?

\( \sqrt{2}, \sqrt{5}, \pi, e, 2 + \sqrt{2}, \ldots \)

Does \( 2^\sqrt{2} \) even make sense? Yes
Find "nice" decimals close to $\sqrt{2}$

$1 \quad 1.4 \quad 1.41 \quad 1.414 \ldots \rightarrow \sqrt{2}$

$2 \quad 2.4 \quad 2.41 \quad 2.414 \ldots \rightarrow 2$

Even if you don't know the exact value, you can still work with them.

Rules

$a^0 = 1$

$a^x \cdot a^y = a^{x+y}$

$\frac{a^x}{a^y} = a^{x-y}$

$a^{-x} = \frac{1}{a^x}$

$(a^x)^y = a^{xy}$

Numerical example

$2^0 = 1$

$2^3 \cdot 2^5 = 2^8$

$\frac{2^3}{2^5} = 2^{-2}$

$2^{-3} = \frac{1}{2^3}$

$(2^3)^5 = 2^{15}$
Exponential function: \( f(x) = a^x \), \( a > 0 \)
\( x \in \mathbb{R} \)
\( \uparrow \) "in"
\( a \neq 1 \)

\[ a = \text{base} \]
\[ x = \text{exponent} \]

Problems: If \( a = 1 \), \( f(x) = 1 \) ! (constant)

If \( a < 0 \), \( f(x) \) is sometimes negative, sometimes positive

Graph is not nice

Polynomials vs exponentials

\[ f(x) = 2^x \quad g(x) = x^2 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
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<tbody>
<tr>
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</table>
$2^x$ ultimately wins over $x^2$

Graph of $2^x$

$x \uparrow, \ 2^x \uparrow$
$x \downarrow, \ 2^x \text{ goes to}$

$\lim_{x \to \infty} f(x) = \infty$
$\lim_{x \to -\infty} f(x) = 0$

Challenge question:
Can you find the intersection pts?}

$\frac{x}{2} = x^2, \quad x = ?$
\[ f(x) = \left(\frac{1}{2}\right)^x \]

\[ \lim_{x \to -\infty} f(x) = 0 \]

\[ \lim_{x \to -\infty} f(x) = \infty \]

\[ \text{In general} \]

\[ f(x) = a^x \]

"Concave up"

\[ a > 1 \]

\[ a = 1 \]

\[ 0 < a < 1 \]
What is $e$?

**FACTS**

* $e \approx 2.71828$

* $2^x \leq e^x \leq 3^x$

* $e$ chosen by Euler in 1727

* $e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n$

* $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$
(a) Find domain and range for \( y = 3 - 2^x \). Sketch it.

\[
\begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & (-\infty, 3) \\
\end{align*}
\]

\[
[0 < 2^x < \infty \implies -\infty < -2^x < 0 \implies -\infty < 3 - 2^x < 3]
\]

(b) Find domain, range for \( y = \frac{1}{2} e^{-x} - 1 \)

\[
\begin{align*}
\text{Domain: } & \mathbb{R} \\
\text{Range: } & (-1, \infty) \\
\end{align*}
\]

\[
\begin{align*}
0 < e^{-x} < \infty \\
0 < \frac{e^{-x}}{2} < \infty \\
-1 < e^{-x/2} - 1 < \infty
\end{align*}
\]