Final Exam - Practice Problems

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25 April 2015

Precal/Calc review and (1.5 -1.6)

1. \(\lim_{x \to \infty} \frac{5x^2 - 4x + 7}{3x^2 - 6} = \)

2. Find the tangent to \(f(x) = \sin 3x + \cos 2x\) at the point \((\pi/6, 1)\).

3. Factorize \(3x^2 - 7x - 6\)

4. Find the derivative of \([\cos(\ln x) - x \sin x + \frac{x+1}{x-1}]\)

5. Find the integral of \([\frac{x^2 + \sec^2 x}{x^3 + 3\tan x} + 2x^7]\)

6. Find the integral of \(\tan x\)

Integration by parts (7.1)

1. \(\int_1^e (x^2 \ln(\sqrt{x})) \, dx\)

2. \(\int (t \sec^2(3t)) \, dt\)

Trigonometric identities and integrals (7.2)

- Read Strategies in Pages 473, 474.
- Formula sheet given before in class and Formula box in Page 476
1. \( \int (\sin 8x \cos 5x) \, dx \)
2. \( \int (\sin^3 x \cos^4 x) \, dx \)
3. \( \int (\tan^2 x + \tan^4 x) \, dx \)

Partial fractions (7.4)

Reading Revision

- Revise polynomial division
- Case I, Example 2 in Pg 486
- Case II, Example 4 in Pg 488

1. \( \int_4^5 \left( \frac{x^3 - 4x - 10}{x^2 - x - 6} \right) \, dx \)
2. \( \int \left( \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} \right) \, dx \)

Approximate integration (7.7)

Reading Revision

- Midpoint rule formula in Page 507
- Error bound for midpoint rule \((E_M)\) on Page 510

Approximate using midpoint formula and find the error bound using two rectangles for

\[
\int_0^\pi x \cos x
\]
Improper integrals (7.8)

Reading Revision

• Type I, Example 2 on Page 521
• Type II, Example 7 on Page 524

Evaluate the following improper integrals [use the language of limits]

1. $\int_0^\infty xe^{-4x} \, dx$
2. $\int_0^1 \ln 2x \, dx$

Arc length and surface area (8.1 and 8.2)

Reading Revision

• Arc length formulae in Pages 539 and 540
• Surface area formulae in Page 547

Set up (but do not evaluate) the following integrals for calculating

1. The arc length of $y = \frac{x^3}{3} + \frac{1}{4x}$ for $1 \leq x \leq 2$
2. The surface area obtained by rotating the curve $y = e^{-x^2}$, $-1 \leq x \leq 1$ about the $x$-axis.

9.1 - 9.5

Separable and linear differential equations

1. $\frac{dy}{dx} = \frac{\ln x}{xy}$ where $y(1) = 2$
2. $\frac{dy}{dt} = \frac{2t}{ey + t^2}$ where $y(0) = 0$
3. $xy' - y = x \ln x$ where $y(1) = 2$ and $x > 0$
Orthogonal trajectories and mixing problems

1. Find the orthogonal trajectories for the family of curves $y = ke^x$ where $k$ is an arbitrary constant.

2. A tank contains 100 litres of pure water. Brine that contains 0.1 kg of salt per liter of water enters the tank at a rate of 10 litre per min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is there in the tank after 6 minutes?

Growth models, direction fields etc

1. Write down the differential equation and general solution for the exponential and logistic growth models.

2. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.1P \left(1 - \frac{P}{2000}\right)$$

where $t$ is measured in weeks.

(a) What is the carrying capacity?

(b) What is the value of $k$?

(c) What is the slope of the arrow at $(0, 2000)$ in the direction field?

(d) What is the slope of the arrow at $(0, 100)$ in the direction field?

(e) If the initial population is 100, what happens to the population in the long run? (i.e., find $\lim_{t\to\infty} P(t)$). Also find the population in 20 weeks.

Polar coordinates and areas (10.3 and 10.4)

1. Find the polar coordinates of a point whose $x, y$ coordinates are given by $(2, -2)$.

2. Find the $x, y$ coordinates of a point whose polar coordinates are given by $(2, \frac{\pi}{4})$.

3. Find the slope of the tangent line to the polar curve given by $r = 2\sin\theta$ at $\theta = \pi/6$. 
4. Find the area of the region bounded by the curve \( r = e^{-\theta/4} \) where \( \pi/2 \leq \theta \leq \pi \)

**Sequences (11.1)**

Find the limits of the following sequences. [Sometimes they might not exist!]

1. \( \left\{ \frac{3n}{1+6n} \right\}_{n=1}^{\infty} \)

2. \( \left\{ 1 + \frac{10^n}{9^n} \right\}_{n=1}^{\infty} \)

3. \( \left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \right\} \)

**Geometric series and Test for divergence (11.2)**

1. Find whether the following series are convergent. If yes, to what do they converge to. If not, justify.

   (a) \( \sum_{n=1}^{\infty} \frac{12}{(-5)^n} \)

   (b) \( \sum_{k=1}^{\infty} \frac{3n}{1+6n} \)

2. Convert 2.516516516516... into a rational number

**Integral Test and estimates (11.3)**

1. State the Integral test.

2. Find whether the following series are convergent. Justify.
3. State the remainder estimate for the Integral test
4. Find an upper bound for $R_{10}$ for $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Comparison tests (11.4)

1. State the Comparison and the Limit comparison tests.
2. Find whether the following series are convergent. Justify.
   
   (a) $\sum_{n=1}^{\infty} \frac{1 + \sin n}{10^n}$

   (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 1}$

Alternating test (11.5)

1. State the Alternating test.
2. Find whether the following series are convergent. Justify.
   
   (a) $\sum_{n=1}^{\infty} (-1)^n \left( \frac{n}{10^n} \right)$

   (b) $\sum_{n=1}^{\infty} (-1)^n \left( \frac{3n - 1}{2n + 1} \right)$
3. State the remainder estimate for the Alternating test

4. Find an upper bound for $R_8$ for $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

**Absolute convergence, ratio and root tests (11.6)**

1. State the Ratio and Root tests

2. Find whether the following series are absolutely convergent. Justify.

   (a) $\sum_{n=1}^{\infty} \left(\frac{\cos n}{n^2}\right)$

   (b) $\sum_{n=1}^{\infty} \left(\frac{n^n}{n!}\right)$

   (c) $\sum_{n=1}^{\infty} \left(\frac{(2n + 3)^n}{(3n + 2)^n}\right)$

**Power series (11.8-11.9)**

(a) Find the power series representation around 0 for $f(x) = \frac{5}{1 - 4x^2}$.
   Find its radius of convergence

(b) Find the power series for $\int f(x) = \int \left(\frac{5}{1 - 4x^2}\right)$

(c) Find the power series representation around 0 for $g(x) = \frac{1}{1 + x^2}$.
   (Hint: Think of a function whose derivative is $g$)

**Taylor series (11.10)**

(a) Find the Maclaurin series for $f(x) = \sqrt{1 + x}$.

(b) Find the Maclaurin series for $f(x) = e^{-2x}$. Find its interval of convergence

(c) Find the Taylor series for $f(x) = \ln x$ around $a = 2$.
Applications (11.11)

1. Find the Taylor polynomial $T_4(x)$ for $f(x) = \sin x$ around $a = \pi/6$

2. Find the error for $0 \leq x \leq \pi/3$ using Taylor’s inequality