Signed areas

Area = +14 sq. units

Area = -14 sq. units

area = (2 \times 5) \quad (-2 \times 2)
= 10 - 4
= +6 sq. units
Estimate 1

\[1 \times 5 + 1 \times 4 = 9 \text{ sq units}\]

Estimate 2

\[1 \times 4 + 1 \times 3 = 7 \text{ sq units}\]
(left Riemann sum)

\[ \Delta x = x_2 - x_1 = x_1 - x_0 \]

\[
\begin{align*}
\sum_{i=0}^{n-1} f(x_i) \Delta x
\end{align*}
\]

(Left Riemann sum)

(right Riemann sum)

\[
\sum_{i=1}^{n} f(x_i) \Delta x
\]

(right Riemann sum)
Both left, right Riemann sums are estimates of actual area.

As \( n \to \infty \), (you divide area into very thin strips), estimate gets better.

Note: sometimes left, right Riemann sum \( \neq \) area

...just an estimate! (sometimes over, sometimes under)

Area under \( f(x) = y \) between \( x = a \) and \( x = b \)

\[
\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
\Delta x = \frac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, x_n = b
\]
Can also define area \( A \) as the limit of a right Riemann sum:

\[ A = \lim_{n \to \infty} \left[ f(x_1) \Delta x + \ldots + f(x_n) \Delta x \right] \]

\[ \Delta x = \frac{b-a}{n} \]

\( \Delta x \)

\( \Delta x \)

\( a \)

\( x_0 \)

\( x_1 \)

\( \ldots \)

\( x_{n-1} \)

\( b \)

\( x_n \)

(Answer coincides!)

\[ \int_{a}^{b} f(x) \, dx := \text{ (stands for) signed area under } y = f(x) \text{ between } x = a \text{ and } x = b \]

\[ = \lim_{n \to \infty} \left( \sum_{i=1}^{n} f(x_i) \Delta x \right) \]

Note limit automatically gives signed area

So if you input 5, 4, 2, 4, 4, 8
So if you input $\int_{-7}^{4} \frac{y}{2} \, dx$, the answer will be -14 square units.

If you input $\int_{2}^{5} \frac{5}{2} \, dx$, the answer will be shaded (area) = +10 square units.
If \( f(x) = x^3 \), find the signed area under \( x^3 = y \) bet \( x = -2 \) and \( x = +2 \)

\[
\int_{-2}^{2} x^3 \, dx = ?
\]

It's going to be difficult to compute Riemann sums...

but we can find the answer easily in this case.

**Key word** "anti-symm"

\[
f(-2) = -f(2) \\
f(-x) = -f(x)
\]

So, right side \( y \)-axis = + a square

Left side \( y \)-axis = - a square

Total signed area = 0
Let us find it by Riemann sums!

(Use right Riemann sum)

\[ \Delta x = \frac{4-a}{n} = \frac{4}{n} \]

\[ x_0 = 0 \]
\[ x_1 = 0 + \frac{4}{n} = \frac{4}{n} \]
\[ x_2 = \frac{8}{n} \]
\[ x_3 = \frac{12}{n} \]
\[ \vdots \]
\[ x_{n-1} = \frac{4(n-1)}{n} = \frac{4n-4}{n} \]
\[ x_n = \frac{4n}{n} = 4 \]
Right Riemann sum:

\[ \text{# of rectangles} \rightarrow (n, x^2, a, b) \uparrow \text{end pts} \]

function

\[ = f(x_1) \Delta x + \cdots + f(x_n) \Delta x \]

\[ = \left( \frac{4}{n} \right)^2 \Delta x + \left( \frac{8}{n} \right)^2 \Delta x + \cdots + \left( \frac{4n}{n} \right)^2 \Delta x \]

\[ = \left( \frac{4}{n} \right)^3 \left( \frac{1}{n} \right) + \left( \frac{8}{n} \right)^2 \left( \frac{1}{n} \right) + \cdots + \left( \frac{4n}{n} \right)^2 \left( \frac{1}{n} \right) \]

\[ = \left( \frac{4}{n} \right)^3 \left[ \sum_{i=1}^{n} i^2 \right] \]

What is \( \sum_{i=1}^{n} i^2 \)?

Formula:

\[ \frac{n(n+1)(2n+1)}{6} \]

\[ \text{Right Riemann sum} = \frac{64}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right] \]

\[ = \frac{32}{3} \left( \frac{n(n+1)(2n+1)}{n^2} \right) \]
\[
\lim_{n \to \infty} \frac{32}{3} \left[ \frac{2n^2 + 2n + n + 1}{n^2} \right]
= \frac{32}{3} \lim_{n \to \infty} \left[ 2 + \frac{3}{n} + \frac{1}{n^2} \right]
= \frac{32}{3} \left( 2 + 0 + 0 \right)
= \frac{32}{3} \times 2
= \frac{64}{3}
\]