**Differential Equations**  

**Example 1:** \( y'' + 2xy^2 y' + 7xy = 0 \)

Double derivative  
Second order

**Example 2:** \( y' = 7x + y \)

First derivative  
First order

**Solving**

\[ y' = f(x) \]

**Theory**

\[ y' = f(x) \]
\[ y = \int f(x) \, dx \]
\[ = q(x) + c \]

Anti-derivative

**Example**

\[ y' = 2x^2 \]
\[ y = \int 2x^2 \, dx \]
\[ = \frac{2x^3}{3} + c \]

"General solution"

**Initial Value Problem**

Solving \( y' = f(x) \)  
\[ y(0) = ? \]

**Example**

Give \( y' = 2x^2 \), \( y(0) = 7 \)  
\[ \Rightarrow y(0) = c = 7 \]

*Particular solution*  
\[ y = 2x^2 + 7 \]
Direction (slope) fields

How to visualize?

\[ y' = \frac{x^2}{4} + c \]

\[ y = \frac{x^2}{4} + c \]

→ At pt (x, y), draw arrows with slope \( y' \)

→ (10) here

(1, 0) → arrow with slope \( \frac{1}{2} \)

(1, y) → “

(-1, 1) → arrow with slope \( -\frac{1}{2} \)

(0, y) → arrow with slope 0

→ Solutions go differential due "flow" along there!

→ Notice no \( x^n \) as different particular solutions (this is a general fact!)

(So initial condition → unique solution!)

for \( y' = f(x, y) \) in a nbhd

Theorem

Consider \( y' = f(x, y) \) \( \{ \) IVP, first order

\( y(x_0) = y_0 \)

Assume \( f, \frac{df}{dy} \) are cont. near \( (x_0, y_0) \)

→ IVP has! solution on some interval including \( x_0 \)

⇒ So different \( f \), \( y' = f(x, y) \) dont \( x^n \)!
\[ \text{acc} \rightarrow \text{vel} \rightarrow \text{position} \]
\[ v(0) \quad s(0) \quad \text{given} \]

\[ y'' = f(x) \quad \rightarrow \quad y' = \int f(x) \, dx \]
\[ y'(0) = * \quad \rightarrow \quad y = \int y'(x) \, dx \]
\[ y(0) = ** \quad \rightarrow \quad \text{solve for } y \]

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Verifying whether \( y = f(x) \) is a solution for DE:

\[ 2x^2 + 7x + 1 = 0 \]

Is \( x = -1 \) a sol? What do you do?

Plug in, check:

\[ 2(1) + 7 + 1 \neq 0 \]

so \(-1\) not a sol

\[ y'' - 4y' + 3y = 0 \]

Similarity:

\[ y'' - 4y' + 3y = 0 \]

Is \( y = 4e^{2x} \) a solution?

\[ y'' = 4e^{2x} \]

\[ y' = 4e^{2x} \]

\[ 4e^{2x} - 16e^{2x} + 12e^{2x} = 0 \]

\[ \rightarrow \text{yes!} \]

\[ y = 4e^{2x} \text{ is a solution.} \]

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Hw: Is \( y = 4e^x + 5e^{3x} \) a solution?

\[ \rightarrow \text{worked out in book} \]
36) \( y' = \frac{y}{2} \)

- slope field sketch
- guess sol from graph
- actually write down solution!

**HW 8.1**

5, 6, 9, 10, 15, 16, 19, 20, 22, 23,
25, 28, 33, 34, 35, 37, 44, 45,

**sec 8.2**

**Linear first order diff eq**

\[
\begin{align*}
y' + p(x) y &= q(x) \\
\text{first order} &\quad \text{linear}
\end{align*}
\]

ex: \( y' + (\mp x^2 + 3) y = 8x + \sin x \)

If \( q(x) = 0 \), \( y' + p(x)y = 0 \) \( \leftrightarrow \) Linear homogeneity

ex: \( y' = -7xy \)

\( \rightarrow \) \( y' + 7xy = 0 \)
Note: $y' + x^2 = e^x < \text{ NOT } \frac{\text{LFODE}}{\text{Linear first order diff eq}}$

$y^2$ term there!

Then:

\[ y' + p(x)y = q(x) \]  \text{ IVP}
\[ y(x_0) = y_0 \]  \text{ Linear first order diff eq}

→ Assume $p, q$ cont on an interval $I$ containing $x_0$

→ Then unique sol to this IVP defined for every pt in $I$ exists.

So if $p, q$ cont everywhere, then sol exists everywhere.

Finding intervals where solutions exist:

1. $y' = 7xy$
   \[ y(0) = 2 \]
   \[ p(x) = -7x \]
   \[ q(x) = 0 \]
   \[ \text{cont all of } \mathbb{R} \]
   \[ (-\infty, \infty) \]

2. $y' - \frac{x}{x^2+1}y = \frac{e^x}{x^2+1}$
   \[ y(0) = 4 \]
\[ p(x) = -\frac{x}{x^2+1} \]
\[ q(x) = \frac{e^x}{x^2+1} \]

so \((-\infty, 0)\)

3) \[ x y' + y = x^3 \quad y(2) = -3 \]

\[ \rightarrow y + \frac{y}{x} = x^2 \quad y(2) = -3 \]

\[ p(x) = \frac{1}{x} \leftarrow (-\infty, 0) \quad (0, \infty) \]

\[ q(x) = x^2 \leftarrow \text{cont everywhere} \]

Need to pick interval containing \(x_0 = 2\)
Solving these

**Theory**

1) Get it in the form
   \[ y' + p(x) \cdot y = q(x) \]

2) Identify \( p(x) \) and \( q(x) \)

3) \( F(x) = \int p(x) \, dx \)

4) \( g(x) = \ln \mid \text{integrand} \mid \)
   \[ g(x) = \ln \left| e^{\int p(x) \, dx} \right| \]
   \[ = e^{\int p(x) \, dx} \]

5) **MAGIC!**

   - Find \( \int g(x) \cdot q(x) \, dx \)
   - Divide by \( g(x) \)

\[ -\frac{1}{3} e^{-x^3} + c \]

\[ e^{-x^3} \]

\[ \frac{-1}{3} + C e^{x^3} \]

\[ \text{Solution} \]

**Example**

1) \[ y' = x^2 + 3x^2y \]
   \[ y' - 3x^2y = x^2 \]

2) \[ p(x) = -3x^2 \]
   \[ q(x) = x^2 \]

3) \[ F(x) = \int -3x^2 \, dx \]
   \[ = -x^3 \]

4) \[ g(x) = e^{-x^2} \]

\[ \int e^{-x^3} \cdot x^2 \, dx = \]

\[ u = -x^3 \]

\[ du = -3x^2 \, dx \]

\[ \Rightarrow -\frac{1}{3} \int e^u \, du \]

\[ \Rightarrow -\frac{1}{3} e^{-x^3} \]

\[ \Rightarrow -\frac{1}{3} e^{-x^3} + c \]

Add a \( C \)
Check!

\[ y = -\frac{1}{3} + ce^{x^3} \]
\[ y' = 3c e^{x^3} x^2 \]
\[ y' - 3x^2 y = 3c e^{x^3} x^2 - 3x^2 \left( -\frac{1}{3} + ce^{x^3} \right) = x^2 \]

\[ y' + p(x) y = q(x) \]
\[ G(x) y' + G(x) p(x) y = G(x) q(x) \]
\[ \left( G(x) y \right)' = G(x) q(x) \]
\[ G(x) y = \int G(x) q(x) \]
\[ y = \frac{\int G(x) q(x)}{G(x)} \]

why doesn't work:

\[ G(x) = e^{F(x)} \]
\[ G'(x) = e^{F(x)} F'(x) \]
\[ G(x) p(x) = e^{F(x)} p(x) \]
\[ \int p = F \]
\[ G(x) p(x) \]

remember to add a c then divide!
Solve \( xy' + 2y = 5x^3 \)

\[ y(1) = 5y \]

1. Bring it to \( y' + p(x)y = q(x) \)
   
   \[ xy' + 2y = 5x^3 \]
   
   \[ y' + \frac{2}{x} y = 5x^2 \]

2. \( p(x) = \frac{2}{x} \)
   
   \[ q(x) = 5x^2 \]

3. \( F(x) = \int p(x) = \int \frac{2}{x} \, dx = 2 \ln |x| = \ln x^2 \)

4. \( G(x) = e^{F(x)} = e^{\ln x^2} \)
   
   \[ = x^2 \]

5. \( \int G(x)q(x) = \int x^2 \cdot 5x^2 \, dx \)
   
   \[ = \int 5x^4 \, dx \]
   
   \[ = \frac{5x^5}{5} + c \]
   
   \[ = x^5 + c \]

6. \[ \frac{\int G(x)q(x)}{G(x)} = \frac{x^5 + c}{x^2} = x^3 + cx^{-2} \]
\[ y = x^3 + \xi x^{-2} \]
\[ y' = 3x^2 + -2Cx^{-3} \]
\[ xy' + 2y = 3x^3 - 2Cx^2 + 2x^3 + 2C x^{-2} \]
\[ = 5x^3 \]

But not yet done

**IVP**

\[ y(1) = \frac{5}{4} \]
\[ y(x) = x^3 + Cx^{-2} \]
\[ y(1) = 1 + C = \frac{5}{4} \]
\[ \therefore C = \frac{1}{4} \]

\[ y(x) = x^3 + \frac{1}{4x^2} \]
One compartment model

Absorption rate

(dotted arrow = not dependent on $Q$
(uptake rate)

Elimination rate
(relative rate)

Ex: Mussel in polluted water with polychloride biphenyls (PCB)

$Q(t) = \text{conc in mussel } \times \text{PCB (mg/gm tissue)}$

after 6 days

Mussels absorb 12 mg PCB/gm tissue/day

Elimination rate = 0.18 $Q$ mg/gm tissue/1 day

$Q' = -0.18Q + 12$
Solve for $Q$!

What happens if no PCB is removed initially?

What happens after a very long time?

$$Q' = -0.18Q + 12$$
$$Q' + 0.18Q = 12$$

$p(t) = 0.18 \quad f(t) = 0.18k \quad g(t) = e^{0.18t}$

$q(t) = 12$

$$Q(t) = \frac{\int g(t) q(t) dt}{g(t)} = \frac{\int 12 e^{0.18t} dt}{e^{0.18t}}$$

$$= \frac{12 e^{0.18t}}{0.18} + c$$

$$= \frac{200}{3} + ce^{-0.18t}$$

$q(0) = 0 = \frac{200}{3} + c$

$$\therefore c = -\frac{200}{3}$$

$q(t) = \frac{200}{3} + \frac{200}{3} e^{-0.18t}$

$t \to \infty / \begin{bmatrix} 200 \cr \frac{200}{3} \end{bmatrix}$
$t \to \infty$

\( \phi \) shd stabilize

\( \phi' = 0 \)

\[ 0 + 0.18 \phi = 12 \]

\[ \phi = \frac{12}{0.18} = \frac{200}{3} \quad \text{as } t \to \infty \]

next semester more at equilibrium!

Read ex 4 do sec 8.2

HW 8.2 3, 5, 13, 15, 17, 31, 20, 37, 38,

39, 40**