This midterm totals 60 points and lasts for 75 minutes. Good luck!

I. Hoppity Grasshopper is travelling in a straight line. He starts off at the origin (initial position = 0) with an initial velocity of 1 cm/s. You are given his acceleration, \( a(t) = (\sqrt{t} + 1) \text{ cm/s}^2 \). Answer the following questions. Remember to put units everywhere!

1. (1 pt) Find his acceleration at 3 seconds.

\[
a(3) = \sqrt{3+1} = 2
\]

Answer: \( 2 \text{ cm/s}^2 \)

2. (2 pts) Find his velocity function. Also find his velocity at 3 seconds.

\[
v(t) = \int \sqrt{t+1} \, dt = \frac{2}{3} (t+1)^{3/2} + C
\]

\[
v(0) = 1 = \frac{2}{3} + C \implies C = \frac{1}{3}
\]

\[
v(t) = \frac{2}{3} (t+1)^{3/2} + \frac{1}{3} \text{ cm/s}
\]

Answers: \( v(t) = \frac{2}{3} (t+1)^{3/2} + \frac{1}{3} \text{ cm/s}, \quad v(3) = \frac{2}{3} \cdot 8 + \frac{1}{3} = \frac{16}{3} \text{ cm/s} \)

3. (2 pts) Find his position function.

\[
s(t) = \int \frac{2}{3} (t+1)^{3/2} + \frac{1}{3} \, dt
\]

\[
= \frac{2}{3} \cdot \frac{2}{5} (t+1)^{5/2} + \frac{1}{3} t + D
\]

\[
= \frac{4}{15} (t+1)^{5/2} + \frac{t}{3} + D
\]

\[
s(0) = 0 = \frac{4}{15} + D
\]

\[
+ D = -\frac{4}{15}
\]

Answer: \( s(t) = \frac{4}{15} (t+1)^{5/2} + \frac{t}{3} - \frac{4}{15} \text{ cm} \)
II. Fill in the blanks

(10 pts) *No steps needed here.*

1. \( \int \cos \left( \frac{\pi}{6} x \right) \, dx = \frac{\sin \left( \frac{\pi}{6} x \right)}{\frac{\pi}{6}} \)

2. A population of mice growing exponentially has a doubling time of \((\ln 16)\) days. Its growth rate is \(25\)\%.

3. \( \int_1^5 (\sin^2 x + \cos^2 x) \, dx = 4 \)

4. \( \frac{d}{dx} \left( \int_x^t \sin t \, dt \right) = \sin x \)

5. \( \lim_{x \to \infty} \frac{x^2 + 1}{e^x} = 0 \)

6. \( \int (\tan 2x \sec 2x) \, dx = \frac{(\sec 2x)^2}{2} \)

7. \( \int_0^5 xe^x \, dx = 0 \)

8. \( \int_1^\frac{1}{2} \, dt = \ln |8| - \ln |1| = 1 \)

9. If \( g \) is a continuous function such that \( g(-x) = -g(x) \), then \( \int_{-2}^2 g(x) \, dx = 0 \)

10. The half life of a radioactive material decaying exponentially is \((\ln 2)\) days. If the initial amount present is 10 grams, then the quantity of radioactive material is given by

\[
Q(t) = 10 e^{-\frac{t}{\ln 2}} \quad k = \frac{\ln 2}{\ln 2} = 1
\]

\[
k = \frac{\ln 2}{\ln 16} = \frac{\ln 2}{4 \ln 2} = \frac{1}{4} = 25\%
\]
III. Rectangles à la Riemann

(6 pts) Recall that $R_{left}(n, f(x), a, b)$ is the left Riemann sum with $n$ rectangles for the function $f(x)$ with start point $x = a$ and end point $x = b$. Similar notation for the right Riemann sum $R_{right}$.

1. (3 pts) Find $R_{left}(3, f(x) = \frac{1}{1+x}, 0, 3)$.

\[
(1) \ f(0) + (1) f(1) + (1) f(2) \\
= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\
= \frac{6 + 3 + 2}{6} \\
= \frac{11}{2}
\]

Answer: $\frac{11}{2}$ square units

2. (3 pts) Find $R_{right}(3, f(x) = \frac{1}{1+x}, 0, 3)$.

\[
(1) f(1) + (1) f(2) + (1) f(3) \\
= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
= \frac{6 + 4 + 3}{12} \\
= \frac{13}{12}
\]

Answer: $\frac{13}{12}$ square units
IV. The fundamentals

1. (2 pts) State the first form of the fundamental theorem of calculus. Define any terms you use in the statement!

2. (2 pts) State the second form of the fundamental theorem of calculus. Define any terms you use in the statement!
Va. Find my age!

(5 pts) I am a skeleton hiding in your family closet and I had some radioactive material $X$ in me just before I died. Only $\frac{1}{8}$-th portion of the original amount remains in me. The half life of the radioactive material $X$ is $(\ln 2^{1000})$ years. Find my age! SHOW STEPS for I am a skeleton hard to convince.

$$k = \frac{\ln 2}{\ln 2^{1000}} = \frac{1}{1000}$$

$$X(t) = C e^{-\frac{t}{1000}}$$

$$= \frac{1}{\sqrt{e}} \ C$$

$$e^{-\ln 2} = e^{-\frac{1}{1000} \ t}$$

$$\frac{1}{2} = \frac{1}{1000} \ t$$

$$500 = t$$

My age is $\boxed{500}$ years.
Vb. Find my temperature!

(5 pts) I am a very hot liquid. At time 0 mts, I am placed in a refrigerator whose thermostat is set at 10°C. At time 70 mts, my temperature is $10 + \frac{180}{e^{70}}^\circ C$ and at time 80 mts, my temperature is $10 + \frac{180}{e^{80}}^\circ C$. Find my initial temperature. Show NEAT WORK.

\[ T(t) = P_0 e^{-kt} + 10 \]

\[ T(70) = P_0 e^{-70k} + 10 = 10 + \frac{180}{e^{70}} \]

\[ \Rightarrow P_0 e^{-70k} = \frac{180}{e^{70}} \]

\[ \Rightarrow P_0 \frac{e^{-70k}}{e^{70k}} = \frac{180}{e^{70}} \]

\[ \Rightarrow \frac{P_0}{e^{80k}} = \frac{180}{e^{80}} \]

(1)

(2)

\[ \frac{\text{1169}}{e^{80k}} = \frac{180}{e^{80}} \]

\[ \Rightarrow e^{10k} = e^{10} \]

\[ \Rightarrow k = 1 \]

\[ \ln(1) \Rightarrow P_0 = 180^\circ C \]

My initial temperature is \[ 180^\circ C \].
VI. Integration

(15 pts) Integrate each of these by using either substitution/parts. State which method you are using and SHOW NEAT WORK. Simplify as much as possible.

1. \[ \int (x + 1)\sqrt{x + 2} \, dx \]

\[ u = x + 1 \quad \quad du = dx \]

\[ dv = \sqrt{x + 2} \, dx \quad \quad v = \frac{2}{3} (x + 2)^{3/2} \]

\[ \int u \, dv = uv - \int v \, du \]

\[ = \frac{2}{3} (x + 1) (x + 2)^{3/2} - \int \frac{2}{3} (x + 2)^{3/2} \, dx \]

\[ = \frac{2}{3} (x + 1) (x + 2)^{3/2} - \frac{4}{15} (x + 2)^{5/2} + C \]

Answer: \[ \frac{2}{3} (x + 1) (x + 2)^{3/2} - \frac{4}{15} (x + 2)^{5/2} + C \]
\[ \int x \sqrt{x^2 + 9} \, dx \]

Let \( u = x^2 + 9 \)
\[ du = 2x \, dx \]

\[ \int \sqrt{u} \frac{du}{2} \]

\[ = \int \frac{u^{1/2}}{2} \, du \]

\[ \rightarrow \frac{1}{2} \frac{2}{3} u^{3/2} \]

\[ \rightarrow \frac{1}{3} u^{3/2} \]

\[ \rightarrow \frac{1}{3} (x^2 + 9)^{3/2} \]

Answer: \[ \frac{1}{3} (x^2 + 9)^{3/2} + C \]
\begin{align*}
\int_0^1 e^{(x+1)} \, dx &= u = x+1 \quad du = dx \\
\quad dv = e^x \, dx \quad v = e^x \\
\int udv &= uv - \int vdu \\
&= (x+1)e^x - \int e^x \, dx \\
&= (x+1)e^x - e^x \\
&= xe^x = g(x) \\
g(1) - g(0) &= 1 \cdot e^1 - 0 \cdot e^0 \\
&= e \\
\text{Answer: } e
\end{align*}
(5 pts) Find whether the following improper integral is convergent or divergent. If it is convergent, find its value. Reason properly using the language of limits etc.

\[ \int_0^\infty \frac{2x}{(1+x^2)^3} \, dx \]

\[ u = (1+x^2)^{-1} \]
\[ du = 2x \, dx \]

\[ \int \frac{du}{u^2} = \frac{-u^{-2}}{-2} \]
\[ = -\frac{1}{2u^2} \]
\[ = -\frac{1}{2(1+x^2)^2} \]

\[ \int_0^b \frac{2x}{(1+x^2)^3} \, dx = \frac{-1}{2(1+b^2)^2} \int_0^b = \frac{-1}{2(1+b^2)^2} + \frac{1}{2} \]

\[ \lim_{b \to \infty} \frac{-1}{2(1+b^2)^2} + \frac{1}{2} = \frac{1}{2} \]

[Convergent]
VIII. Measuring areas

(5 pts) Find the area between \( y = x + 1 \) and \( y = 5x^2 + x - 4 \).

\[
\begin{align*}
  y &= x + 1 \\
  y &= 5x^2 + x - 4 \\
  x + 1 &= 5x^2 + x - 4 \\
  5 &= 5x^2 \\
  x &= \pm 1
\end{align*}
\]

\[
\int_{-1}^{1} (x + 1) - (5x^2 + x - 4) \, dx
\]

\[
= \int_{-1}^{1} (5 - 5x^2) \, dx
\]

\[
= \left[ 5x - \frac{5x^3}{3} \right]_{-1}^{1}
\]

\[
= \left( 5 - \frac{5}{3} \right) - \left( -5 - \frac{-5}{3} \right)
\]

\[
= 5 - \frac{5}{3} + 5 - \frac{5}{3} = 10 - \frac{10}{3} = \frac{20}{3} \text{ square units}
\]