The Final Review
(The One where we realize just how much calculus we ought to have learnt over the semester)

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Last minute advice

• Show steps. Show steps. Show steps.
• The exam starts at eight in the morning. DO NOT stay up late the night before it.

Pre calculus - The One where we try hard to remember the unit circle

1. \[
\frac{\tan \left( \frac{\pi}{4} \right) + \sec \left( \frac{\pi}{4} \right)}{\cot \left( \frac{\pi}{4} \right) + \cosec \left( \frac{\pi}{4} \right)} = \]

2. \[
e^{-3\ln \sqrt{10}} + \frac{\ln 0.1}{\ln 0.0001} = \]

3. \[
\frac{9 + \sin^2 x + \cos^2 x}{9 - \cosec^2 x + \cot^2 x} = \]

4. An interval containing \( x = 5 \) where \( p(x) = \frac{1}{x-3} \) and \( q(x) = \sqrt{x^2 - 9} \) are both continuous?

Limits - The One where we resist the temptation to write \( \frac{0}{0} \)

1. \[
\lim_{x \to -\infty} e^{x^2} x = \]

2. \[
\lim_{x \to 0} \left( \frac{\sin x}{x} \right) = \]

3. \[
\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x^2 - 3x - 10} = \]

4. \[
\lim_{x \to \infty} \frac{3x^2 + 5x - 2}{x^2 - 3x - 10} = \]

5. \[
\lim_{x \to 0} \left( \frac{\sin x}{x} \right) = \]
Continuity - The One where we draw graphs without lifting pen from paper

\( f(x) = [x] \) where \([x]\) is the biggest integer less than or equal to \(x\). Graph \( f(x) \)

1. \( f(1.67) = \) __________
2. \( \lim_{x \to 1.67^+} f(x) = \) __________
3. \( \lim_{x \to 1.67^-} f(x) = \) __________
4. Is \( f \) continuous at \( x = 1.67 \)? Justify.
5. Play the same game for \( x = 2 \). Does your answer differ?

Difference quotients - The One where we are excessively fond of \( \frac{f(x+h)-f(x)}{h} \)

Find \( \frac{d}{dx} \left( \frac{x}{1-x} \right) \) from first principles using the limit of difference quotients.

Differentiation - The One where we obey the rules to derive

Find the derivatives of the following functions. The answers are not particularly nice but then everything is not nice always.

1. \( p(x) = \sin(\cos(x^2)) \)
2. \( p(x) = \frac{\tan^2 x}{\cos x - x} \)
3. \( p(x) = x\sqrt{x+5} \)
4. \( p(x) = \frac{x}{5-x} \)
5. \( p(x) = 2^x + 7x^9 + \sec(\ln x^2) + \tan(2x) \)

Slopes - The One with all the tangents

Find the equation of the tangent to the function \( f(x) = e^x \sin x + 1 \) at the point \((0, 1)\).

Maxima minima - The One where we critically examine our highs and lows

1. Is \( f(x) = x^{2015} + 2015 \) increasing or decreasing on the interval \((-2015, -1)\)?
2. Find all the relative maxima, minima and points of inflection for \( f(x) = \frac{-8x}{x^2+1} \).
3. Find all the absolute maxima and minima for \( f(x) = x^2 - 10x + 8 \) for the interval \([-2, 6]\).
4. Find all the absolute maxima and minima for \( f(x) = x^2 - \frac{2}{x} \) for the interval \((-\infty, 0)\).

Optimization - The One where we do WORD PROBLEMS!

1. An apple farm yields an average of 30 bushels of apples per tree when 20 trees are planted on an acre of ground. Each time 1 more tree is planted per acre, the yield decreases 1 bushel per tree due to the extra congestion. How many trees should be planted in order to get the highest yield?
2. Of all the numbers whose difference is 8, find the two that have the minimum product.
Growth & Decay - The One with the déjá vu

1. An experiment done some years ago reported that a lake contained 5 kg of radioactive material which decays exponentially with a half life of \((\ln 32)\) years. This year (2014), another experiment confirmed that only \(\frac{5}{32}\) kg of the radioactive material remain in the lake. In which year was the first experiment carried out?

2. I am a very hot liquid. At time 0 mts, I am placed in a refrigerator whose thermostat is set at 10\(^\circ\)C. At time 70 mts, my temperature is \(10 + \frac{180}{e^{70}}\)°C and at time 80 mts, my temperature is \(10 + \frac{180}{e^{80}}\)°C. Find my initial temperature.

3. Find the doubling time of bacteria X whose population function on the \(t\)-th day is given by \(P(t) = 2000e^{30t}\) bacteria. What is \(\frac{dP}{dt}\)?

Fundamentals - The One with our favourite theorems of Calculus

1. State both the forms of the fundamental theorem of calculus

2. \[ \frac{d}{dx} \left( \int_2^x \frac{t-1}{t^2+1} \, dt \right) = \]

3. \[ \int_{e^3}^{e^3} \ln x \, dx = \]

Riemann sums - The One where we like drawing rectangles

1. Find \(R_{left}(2, f(x) = \sin x, 0, \frac{\pi}{2})\)

2. Find \(R_{right}(4, f(x) = x^2 - 1, 1, 2)\)

Integration - The One where we integrate and show NEAT STEPS

1. \[ \int_0^1 \frac{24t^5}{4t^6 + 3} \, dt \]

2. \[ \int_0^{e^2} \ln x^7 \, dx \]

3. \[ \int_0^1 (e^x + x) \, dx \]

4. \[ \int t^7(t^8 + 3)^{11} \, dt \]

5. \[ \int 3xe^{3x} \, dx \]

6. \[ \int \tan(3x + 7) \, dx \]

7. Find the area of the region bounded by \(y = x, y = x^5, x = 0\) and \(x = 1\).
Improper integrals - The One where it all converges or diverges

1. \[ \int_{0}^{\infty} x e^{-2x} \, dx \]

2. \[ \int_{1}^{\infty} e^{4x} \, dx \]

Direction fields - The One with all those little pretty arrows

You are given \( y' = \frac{2}{3} x + y \).

1. Give two points \( (x_0, y_0) \) and \( (x_1, y_1) \) where the direction field arrow you draw is horizontal.

2. Find the slope of the direction field at \( (-6, 3) \). If you were asked to pick the direction in which the arrow is pointing, which direction would you pick amongst (N, NE, E, SE, S, SW, W, NW) ?

3. Draw the direction field diagram for this differential equation (draw arrows at all the points \( (x, y) \) where \( x, y \) are integers in \([-2, 2]\))

Differential equations - The One where we solve them

Solve for \( y \) (if an explicit solution is not possible, define it implicitly) which satisfies

1. \( y' = \frac{2}{3} x + y \) and \( y(0) = 4 \)

2. \( y' = e^{2x}(e^{2x} - 2)^4 \)

3. \( x^3 y' + 2x^2 y = 1 \) and \( y(1) = 7 \)

4. \( y' = \frac{x+1}{y} \) and \( y(2) = 4 \)

5. \( y' = \frac{1}{e^{y} + 1} \) and \( y(0) = -1 \).

One compartment models - The One where we do WORD PROBLEMS ...AGAIN!

A tank initially contains 100 lb of salt dissolved in 800 gal of water. Salt water containing 1 lb of salt per gal enters the tank at the rate of 4 gal per minute. The mixture (kept uniform by stirring) is removed at the same rate. How many pounds of salt are in the tank after 2 hrs ?

“To meet with joy and part with thought,
Of teachers and taught, this is the art.”