INTEGRALITY PROPERTIES OF THE CM-VALUES OF CERTAIN WEAK MAASS FORMS

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Abstract. In a recent paper, Bruinier and Ono prove that the coefficients of certain weight $-1/2$ harmonic Maass forms are traces of singular moduli for weak Maass forms. In particular, for the partition function $p(n)$, they prove that

$$p(n) = \frac{1}{24n-1} \cdot \sum P_p(\alpha_Q),$$

where $P_p$ is a weak Maass form and $\alpha_Q$ ranges over a finite set of discriminant $-24n + 1$ CM points. Moreover, they show that $6 \cdot (24n - 1) \cdot P_p(\alpha_Q)$ is always an algebraic integer, and they conjecture that $(24n - 1) \cdot P_p(\alpha_Q)$ is always an algebraic integer. Here we prove a general theorem which implies this conjecture as a corollary.

1. Introduction and Statement of Results

A partition of a positive integer $n$ is any nonincreasing sequence of positive integers which sum to $n$. The partition function $p(n)$, which counts the number of partitions of $n$, is an important function in number theory whose study has a long history. One of the celebrated results of Hardy and Ramanujan on this function, giving rise to the “circle” method, quantifies the growth rate:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$ 

This asymptotic and its method of proof were later refined by Rademacher, yielding an “exact” formula in terms of a modified Bessel function of the first kind $I_{3/2}(\cdot)$ and a Kloosterman sum $A_k(n)$:

$$p(n) = 2\pi(24n - 1)^{-\frac{3}{2}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{3/2} \left( \frac{\pi\sqrt{24n - 1}}{6k} \right).$$

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One can compute values of $p(n)$ from this formula by using sufficiently accurate truncations. Bounding the resulting error is a well-known difficult problem; the best-known bounds are due to Folsom and Masri [4].

In recent work [2], Bruinier and Ono prove a new formula for $p(n)$ as a finite sum of algebraic numbers. These numbers are singular moduli for a weak Maass form which they describe in terms of Dedekind’s eta function and the quasimodular Eisenstein series $E_2$, which are defined in terms of $q := e^{2\pi iz}$ as

$$
\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{and} \quad E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \frac{\sum_{d|n} d q^n}{\eta(n)^2}.
$$

They then define the $\Gamma_0(6)$ weight $-2$ meromorphic modular form:

$$
F_p(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2 \eta(2z)^2 \eta(3z)^2 \eta(6z)^2} = q^{-1} - 10 - 29q - \cdots
$$

Using the convention that $z := x + iy$, with $x, y \in \mathbb{R}$, they define the weak Maass form:

$$
P_p(z) := - \left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F_p(z) = \left( 1 - \frac{1}{2\pi y} \right) q^{-1} + \frac{5}{\pi y} + \left( 29 + \frac{29}{2\pi y} q \right) + \cdots
$$

Bruinier and Ono give a formula for $p(n)$ in terms of discriminant $-24n+1 = b^2 - 4ac$ positive definite integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ satisfying the condition $6 \mid a$. The group $\Gamma_0(6)$ acts on such forms, and we let $\mathcal{Q}_n$ be any set of representatives of those equivalence classes with $a > 0$ and $b \equiv 1 \pmod{12}$. To each such $Q$, we associate the CM point $\alpha_Q$ defined to be the root of $Q(x, 1) = 0$ lying in the upper half of the complex plane. Then the formula of Bruinier and Ono states:

$$
p(n) = \frac{1}{24n-1} \cdot \sum_{Q \in \mathcal{Q}_n} P_p(\alpha_Q).
$$

They further prove that each $6 \cdot (24n-1) \cdot P_p(\alpha_Q)$ is an algebraic integer. They also show that the numbers $P(\alpha_Q)$, as $Q$ varies over $\mathcal{Q}_n$, form a multiset which is a union of Galois orbits for the discriminant $-24n+1$ ring class field. Based on numerics, they made the following conjecture:

**Conjecture** (Bruinier and Ono [2]). *For the Maass form $P_p(z)$ above and for the $\alpha_Q$ in the formula for $p(n)$, we have that $6 \cdot (24n-1) \cdot P_p(\alpha_Q)$ is an algebraic integer.*

Such integrality questions for singular moduli are first mentioned in Zagier’s seminal paper on traces of singular moduli [8], where he notes that singular moduli of non-holomorphic modular functions need not be algebraic integers. The integrality of traces of singular moduli of derivatives of modular forms is studied in depth in
level 1 by Duke and Jenkins \cite{duke}, and further integrality results for other coefficients of Hilbert class polynomials are derived in a forthcoming paper by the second author and Griffin \cite{griffin}.

In this paper, we prove the conjecture of Bruinier and Ono. In fact, we prove that this is the true for all the CM points of discriminant $-24n + 1$ of a wider class of Maass forms. We first recall the following classical Eisenstein series:

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^{n}, \quad E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sum_{d|n} d^5 q^{n}.$$ 

Then our main result is the following:

**Theorem 1.1.** Suppose $F$ is a weakly holomorphic, weight $-2$ modular form on a congruence subgroup such that the Fourier expansions of

$$F \quad \text{and} \quad q \frac{dF}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4}$$

at all cusps have coefficients that are algebraic integers. Let $\alpha_Q$ be the CM point in $\mathbb{H}$ corresponding to a quadratic form $Q(x, y)$ of discriminant $-24n + 1$, and let $P(z)$ be the weak Maass form

$$P(z) = - \left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z).$$

Then $(24n - 1) \cdot P(\alpha_Q)$ is an algebraic integer.

**Remark.** We recall that a meromorphic modular form is said to be weakly holomorphic if its poles are supported on the cusps.

The form $F_p(z)$ studied by Bruinier and Ono satisfies these conditions. One can see this because $F_p(z)$ has level 6, so the group of Atkin-Lehner involutions acts transitively on the cusps. Since $F_p(z)$ is an eigenform for all of the Atkin-Lehner involutions and has an integral Fourier expansion at infinity, it follows that the Fourier expansions of $F_p$ at all cusps is integral. Moreover, since the Atkin-Lehner involutions commute with the Maass raising operator

$$R_{-2} = -4\pi q \frac{d}{dq} - \frac{2}{y},$$

the Fourier expansion of

$$q \frac{dF_p}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4} = F_p \cdot \left( \frac{E_2 - \frac{3}{\pi \mathrm{Im} z}}{\pi \mathrm{Im} z} \right) \frac{E_4 - E_6}{6E_4} - \frac{1}{4\pi} R_{-2} F_p$$

at all cusps is integral as well. Therefore, Theorem 1.1 implies the following:

**Corollary 1.2.** The conjecture of Bruinier and Ono is true.
Remark. Corollary 1.2 is sharp for small (and possibly all) $n$. For example, we have
\[ \prod_{m=1}^{3} (x - P_p(\alpha_{Q_m})) = x^3 - 23x^2 + \frac{3592}{23}x - 419, \]
where $Q_m$ ranges over any choice of representatives of $Q_1$.

Returning to the general case of $P(z)$ as in Theorem 1.1, the work of Bruinier and Ono (Theorem 4.5 of [2]) implies $6 \cdot (24n - 1) \cdot P(\alpha_Q)$ is an algebraic integer. Although this theorem is stated for squarefree level and when $F$ is an eigenfunction of the Atkin-Lehner involutions, an inspection of the proof shows that the assumptions in the statement of Theorem 1.1 are also sufficient (as we are assuming integrality of $f$ at all cusps). Thus it suffices to show that $P(\alpha_Q)$ is integral at primes lying over 6. We will henceforth refer to this property as $6$-integrality. For this purpose, it is convenient to decompose $P$ as
\[ P = A + B \cdot C, \]
where
\[ A = F \left( \frac{dF}{dq} - \frac{1}{6} FE_2 + \frac{FE_6(7j - 6912)}{6E_4(j - 1728)} \right), \]
\[ B = \frac{FE_6j}{E_4}, \]
\[ C = \frac{E_4}{6E_6j} \left( E_2 - \frac{3}{\pi \text{Im} z} \right) - \frac{7j - 6912}{6j(j - 1728)}. \]

To establish the 6-integrality of $P(\alpha_Q)$, it suffices to establish the 6-integrality of each of $A(\alpha_Q)$, $B(\alpha_Q)$ and $C(\alpha_Q)$. In Section 2 we will use methods similar to those of [2] to show that $A(\alpha_Q)$ and $B(\alpha_Q)$ are 6-integral. Then in Section 3 we show that $C(\alpha_Q)$ is 6-integral using a description of $C$ in terms of classical modular polynomials due to Masser.

Remark. For the remainder of this paper, we fix $D \equiv 1 \mod 24$ with $D < 0$, and we let $\alpha_Q$ denote any CM point of discriminant $D$.

2. Proof of 6-Integrality of $A$ and $B$

In this section, we prove the 6-integrality of $A$ and $B$ at the CM-points $\alpha_Q$. We begin by showing that $j(\alpha_Q)$ is a unit at 2 and 3. Recall that in our case, the discriminant is congruent to $-1 \mod 24$, so in particular it is coprime to 2 and 3.

Lemma 2.1. Let $p \in \{2, 3\}$ and $E$ be an elliptic curve defined over a number field $K$. If $E$ has good ordinary reduction at all primes lying over $p$, then $j(E)$ is coprime to $p$. 
Proof. Assume to the contrary that \( j(E) \) was not coprime to \( p \); write \( p \) for a prime ideal lying over \( p \) containing \( j(E) \), and write \( k \) for the residue field \( \mathcal{O}_K/p \).

When \( p = 2 \), the elliptic curve \( E^2 = \mathbb{C}/\mathbb{Z}[\omega] \) (where \( \omega \) is a primitive cube root of unity) has good supersingular reduction at \( p \). But \( j(\omega) = 0 \), so \( E^2/k \cong E_{/k} \), so \( E_{/k} \) is supersingular, which is a contradiction.

Similarly, when \( p = 3 \), the elliptic curve \( E^3 = \mathbb{C}/\mathbb{Z}[i] \) has good supersingular reduction at \( p \). But \( j(\omega) = 1728 \), so \( E^3/k \cong E_{/k} \), so \( E_{/k} \) is supersingular, which is a contradiction. \( \square \)

By this lemma, it suffices to show that both \( B \) and \( A' := A \cdot j \cdot (j - 1728) \) assume integral values at all CM-points.

**Lemma 2.2.** The modular functions \( A' \) and \( B \) are weakly holomorphic and have integral Fourier expansions at all cusps.

Proof. By definition, we have

\[
B = F \cdot E_6 \cdot \frac{j}{E_4},
\]

and by direct examination, all three of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps. Similarly, by definition, we have

\[
A' = F \cdot E_6(j - 864) \cdot \frac{j}{E_4} - (j - 1728) \cdot \left[ j \cdot \left( q \frac{dF}{dq} + F \cdot \frac{E_2E_4 - E_6}{6E_4} \right) \right],
\]

and all of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps. \( \square \)

**Lemma 2.3.** A weakly holomorphic modular function \( g \) for a congruence subgroup \( \Gamma_g \) that has integral Fourier expansions at all cusps is integral at any CM-point.

Proof. (The following argument is classical, probably originally due to Kronecker and Weber.) We consider the polynomial

\[
\Psi_g(X, z) = \prod_{\gamma \in \Gamma_g \setminus \Gamma(1)} (X - g(\gamma z)).
\]

This is a monic polynomial in \( X \) of degree \( [\Gamma(1) : \Gamma_g] \) whose coefficients are weakly holomorphic modular functions in \( z \) for the group \( \Gamma(1) \), so \( \Psi_g(X, z) \in \mathbb{C}[j(z), X] \).

Our assumption that \( g \) has integral Fourier expansion at all cusps implies that for any \( \gamma \in \Gamma(1) \), the modular function \( g \mid \gamma \) has a Fourier expansion at infinity whose coefficients are algebraic integers. Thus, the coefficients of \( \Psi_g(X, z) \) are polynomials in \( j(z) \) whose coefficients are algebraic integers.

Since \( j \) is integral at any CM-point \( \alpha \), the value \( g(\alpha) \) satisfies a monic polynomial whose coefficients are algebraic integers, and is therefore an algebraic integer (note that \( \Psi_g(g(z), z) = 0 \)). \( \square \)
Remark. In Appendix A, we give values of the polynomials $\Psi_A$ and $\Psi_B$ for the form $F_p$ considered by Bruinier and Ono, thus providing a direct proof of the integrality of $A'(\alpha_Q)$ and $B(\alpha_Q)$ in this case.

**Lemma 2.4.** If $\alpha_Q$ is an CM-point with discriminant $D \equiv 1 \mod 24$, then $A(\alpha_Q)$ and $B(\alpha_Q)$ are 6-integral.

**Proof.** This follows from combining Lemmas 2.1, 2.2, and 2.3. □

### 3. Proof of 6-Integrality for $C(\alpha_Q)$

In this section, we finish the proof of Theorem 1.1 by showing that $C(\alpha_Q)$ is 6-integral at the required CM points $\alpha_Q$. To do this, we study the classical modular polynomials $\Phi_{-D}$, in a fashion similar to Appendix 1 of [6]. We begin by reviewing the definition of $\Phi_{-D}$.

**Definition 1.** We say that two matrices $B_1$ and $B_2$ are equivalent if $B_1 = X \cdot B_2$ for some $X \in \text{SL}_2(\mathbb{Z})$.

It is well-known that there are only finitely many equivalence classes of primitive integer matrices of determinant $-D$. Write $M_1, M_2, \ldots, M_n$ for these equivalence classes and suppose $M_1$ is such that $\alpha_Q = M_1 \alpha_Q$.

**Definition 2.** We write $\Phi_{-D}(X, Y)$ for the classical modular polynomial, i.e. the polynomial such that

$$\Phi_{-D}(j(z), Y) = \prod_{i=1}^{n} (Y - j(M_i z)).$$

By [1], Theorem 1 of Section 3.4, the polynomial $\Phi_{-D}(X, Y)$ is symmetric in $X$ and $Y$ and has coefficients that are rational integers. In particular, we can expand $\Phi_{-D}(X, Y)$ in a power series about $X = Y = j(\alpha_Q)$ as

$$\Phi(X, Y) = \sum_{\mu, \nu} \beta_{\mu, \nu} (X - j(\alpha_Q))^\mu (Y - j(\alpha_Q))^\nu,$$

where $\beta_{\mu, \nu} = \beta_{\nu, \mu}$. We write $\beta = \beta_{0,1} = \beta_{1,0}$.

We define $Q$ to be special if there is more than one equivalence class of matrices $M$ such that $M \alpha_Q = \alpha_Q$. This can only happen if $D = 3d^2$ for some integer $d$ (see [6], Appendix 1), so in particular forms of discriminant $-24n + 1$ are not special.

**Lemma 3.1** (Masser). If $Q$ is not special, we have $\beta \neq 0$ and

$$C(\alpha_Q) = \frac{\beta_{0,2} - \beta_{1,1} + \beta_{2,0}}{\beta}.$$
Proof. See [6], Appendix 1 (in particular, equations (100) and (106), and the definition of $\gamma$ on page 118). □

By definition, the $\beta_{\mu,\nu}$ are algebraic integers. Thus, to prove that $C(\alpha_Q)$ is integral at primes lying over 6, it suffices to show that $\beta$ is a unit at primes lying over 6. From the definition of $\beta$, we have

$$\beta = \prod_{i=2}^{n} (j(\alpha_Q) - j(M_i\alpha_Q)).$$

Thus, it suffices to show that for any prime $p$ lying over 6, we have $j(\alpha_Q) \not\equiv j(M_i\alpha_Q) \mod p$. To show this, it is enough to establish another lemma on elliptic curves. Recall that by definition of the $M_i$, we have that $j(\alpha_q) \not\equiv j(M_i\alpha_q)$ for $i = 2, \ldots, n$.

Suppose two such values were congruent, so we would have an isomorphism between the corresponding elliptic curves when reduced mod $p$. Then we show that the isomorphism could be lifted to an isomorphism of the original curves, which is a contradiction.

Lemma 3.2. Suppose $p$ is a prime ideal of a number field $K$. Suppose $E$ and $E'$ are two elliptic curves over $K$ having complex multiplication by orders containing a common order $R$ in a quadratic field $F$. Suppose the index $[O_F : R]$ is coprime to the residue characteristic of $p$. If both curves have good ordinary reduction at $p$ and the reduced curves are isomorphic, then $E$ and $E'$ are also isomorphic.

Proof. Write $k$ for the residue field $O_K/p$ and $p = \text{char}(k)$. As the index of $R$ in $O_F$ is coprime to $p$, there is an isogeny $f : E \to E'$ defined over $\mathbb{C}$ whose degree is coprime to $p$. (Viewing $E$ and $E'$ as quotients of $\mathbb{C}$ by lattices, this follows from the fact that every element of the class group has a representative coprime to $p$.) Enlarging $K$ if necessary, we may suppose $f$ is defined over $K$. Since $E$ has ordinary reduction at $p$, its endomorphisms over $k$ are a rank-2 submodule $S$ of $O_F$ which contains $R$. As the index of $R$ in $O_F$ is coprime to $p$, the index $d$ of $R$ in $S$ is also coprime to $p$. Choose an isomorphism between the reductions $E/k$ and $E'/k$. Composing this with the isogeny $f$ gives an endomorphism of $E/k$, and multiplying this endomorphism by $d$ gives an endomorphism which lifts to an endomorphism $g$ of $E$ whose degree is coprime to $p$. Now the specializations of the kernels of $f \circ d$ and $g$ coincide by construction, and both kernels are subgroups whose order is coprime to $p$. Thus, $\ker f \circ d = \ker g$, and therefore $E \cong E'$.

This completes the proof of the 6-integrality of $C(\alpha_Q)$, as the assumption $D \equiv 1 \mod 24$ shows that the conditions of the above lemma are satisfied. By the discussion in Section [11] this establishes Theorem [1.1].
Appendix A. The Polynomials $\Psi_{A'}$ and $\Psi_B$ for $F = F_p$

Here, we give the explicit values of the polynomials $\Psi_{A'}$ and $\Psi_B$ when $F = F_p$ is the form considered by Bruinier and Ono in [2]. Namely, we have

$$\Psi_{A'} = X^{12} + \sum_{i=0}^{11} a_i X^i \quad \text{and} \quad \Psi_B = X^{12} + \sum_{i=0}^{11} b_i X^i,$$

where the $a_i$ and $b_i$ are the polynomials in $j$ with integer coefficients given below.

\[\begin{align*}
a_{11} &= -2 \cdot (j - 2^6 \cdot 3^3) \cdot (j - 2^5 \cdot 3^3) \cdot j \\
a_{10} &= -(j - 2^6 \cdot 3^3) \cdot j^2 \cdot (7 \cdot 67 \cdot j^2 - 2^6 \cdot 3^2 \cdot 2053 \cdot j + 2^{11} \cdot 3^5 \cdot 31 \cdot 53) \\
a_9 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \cdot (3^2 \cdot j^4 - 2^3 \cdot 6379 \cdot j^3 + 2^6 \cdot 3^2 \cdot 162713 \cdot j^2 \\
&\quad - 2^{12} \cdot 3^3 \cdot 72979 \cdot j + 2^{25} \cdot 3^{12}) \\
a_8 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \cdot (2 \cdot 7 \cdot 13^2 \cdot j^5 - 2^3 \cdot 409 \cdot 3373 \cdot j^4 \\
&\quad + 2^7 \cdot 3^4 \cdot 1237 \cdot 1973 \cdot j^3 - 2^{14} \cdot 3^7 \cdot 5 \cdot 311 \cdot 443 \cdot j^2 \\
&\quad + 2^{41} \cdot 3^{10} \cdot 31 \cdot 2897 \cdot j - 2^{31} \cdot 3^{14} \cdot 163) \\
a_7 &= 2^2 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (11 \cdot 61 \cdot 193 \cdot j^5 - 2^3 \cdot 3 \cdot 27510443 \cdot j^4 \\
&\quad + 2^9 \cdot 3^3 \cdot 97550587 \cdot j^3 - 2^{16} \cdot 3^6 \cdot 11 \cdot 2599451 \cdot j^2 \\
&\quad + 2^{13} \cdot 3^9 \cdot 5 \cdot 739 \cdot 1109 \cdot j - 2^{34} \cdot 3^{13} \cdot 4691) \\
a_6 &= 2^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (2^4 \cdot 3^2 \cdot j^8 + 7 \cdot 199 \cdot 1373 \cdot j^7 \\
&\quad - 2^2 \cdot 29 \cdot 37 \cdot 281 \cdot 13913 \cdot j^6 + 2^{13} \cdot 3^3 \cdot 7 \cdot 233 \cdot 143281 \cdot j^5 \\
&\quad - 2^{15} \cdot 3^2 \cdot 5 \cdot 11 \cdot 21117827 \cdot j^4 + 2^{23} \cdot 3^9 \cdot 3943 \cdot 117577 \cdot j^3 \\
&\quad - 2^{31} \cdot 3^{12} \cdot 769 \cdot 45317 \cdot j^2 + 2^{41} \cdot 3^{16} \cdot 7 \cdot 15923 \cdot j - 2^{50} \cdot 3^{20} \cdot 269) \\
a_5 &= 2^4 \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5 \cdot (2^6 \cdot 3^4 \cdot 5 \cdot j^8 - 7 \cdot 5051 \cdot 5939 \cdot j^7 \\
&\quad + 2^3 \cdot 3^2 \cdot 5 \cdot 61 \cdot 101 \cdot 330037 \cdot j^6 - 2^9 \cdot 3^5 \cdot 96289 \cdot 119173 \cdot j^5 \\
&\quad + 2^{16} \cdot 3^9 \cdot 17 \cdot 7752741 \cdot j^4 - 2^{22} \cdot 3^{11} \cdot 11 \cdot 71 \cdot 523 \cdot 4091 \cdot j^3 \\
&\quad + 2^{35} \cdot 3^{14} \cdot 5 \cdot 673 \cdot 977 \cdot j^2 - 2^{41} \cdot 3^{18} \cdot 79 \cdot 1831 \cdot j + 2^{55} \cdot 3^{24}) \\
a_4 &= (j - 2^6 \cdot 3^3)^4 \cdot j^6 \cdot (2^8 \cdot 3^3 \cdot 5 \cdot 2003 \cdot j^9 - 409 \cdot 39157 \cdot 44483 \cdot j^8 \\
&\quad + 2^9 \cdot 3 \cdot 2092618568983 \cdot j^7 - 2^{20} \cdot 3^4 \cdot 98512996093 \cdot j^6 \\
&\quad + 2^{20} \cdot 3^7 \cdot 41 \cdot 242261 \cdot 608831 \cdot j^5 - 2^{28} \cdot 3^{10} \cdot 5 \cdot 1231 \cdot 155631757 \cdot j^4 \\
&\quad + 2^{32} \cdot 3^{13} \cdot 521 \cdot 3077579657 \cdot j^3 - 2^{42} \cdot 3^{16} \cdot 997 \cdot 1607 \cdot 16657 \cdot j^2 \\
&\quad + 2^{52} \cdot 3^{20} \cdot 23 \cdot 541 \cdot 6863 \cdot j - 2^{63} \cdot 3^{24} \cdot 5 \cdot 11987)
\[ a_3 = 2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j \cdot (3^2 \cdot 377732207 \cdot j^{10} - 2^6 \cdot 5^2 \cdot 7 \cdot 101 \cdot 28520381 \cdot j^9 \\
+ 2^{11} \cdot 11 \cdot 337 \cdot 17990477821 \cdot j^8 - 2^{20} \cdot 3^3 \cdot 179 \cdot 389 \cdot 171956657 \cdot j^7 \\
+ 2^{23} \cdot 3^6 \cdot 5 \cdot 479 \cdot 37193046587 \cdot j^6 - 2^{30} \cdot 3^9 \cdot 1283 \cdot 28703 \cdot 758137 \cdot j^5 \\
+ 2^{36} \cdot 3^{12} \cdot 7 \cdot 31 \cdot 54791988203 \cdot j^4 - 2^{45} \cdot 3^{15} \cdot 192 \cdot 151 \cdot 7738067 \cdot j^3 \\
+ 2^{55} \cdot 3^{20} \cdot 41 \cdot 12810583 \cdot j^2 - 2^{65} \cdot 3^{24} \cdot 1103107 \cdot j + 2^{76} \cdot 3^{27} \cdot 1447) \\
\]
\[ a_2 = 2^2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j \cdot (42967 \cdot 2406947 \cdot j^{11} - 2^3 \cdot 557 \cdot 1783 \cdot 140768209 \cdot j^{10} \\
+ 2^9 \cdot 3^4 \cdot 6205891 \cdot 21226039 \cdot j^9 - 2^{19} \cdot 3^7 \cdot 5 \cdot 11 \cdot 251872948013 \cdot j^8 \\
+ 2^{24} \cdot 3^9 \cdot 5 \cdot 13 \cdot 23 \cdot 37 \cdot 521 \cdot 3203149 \cdot j^7 - 2^{29} \cdot 3^{13} \cdot 47242981376477 \cdot j^6 \\
+ 2^{35} \cdot 3^{16} \cdot 227 \cdot 112292655271 \cdot j^5 - 2^{41} \cdot 3^{18} \cdot 107 \cdot 269749728667 \cdot j^4 \\
+ 2^{54} \cdot 3^{22} \cdot 43 \cdot 449215127 \cdot j^3 - 2^{61} \cdot 3^{27} \cdot 5 \cdot 653 \cdot 54193 \cdot j^2 \\
+ 2^{72} \cdot 3^{30} \cdot 139 \cdot 3719 \cdot j - 2^{82} \cdot 3^{35} \cdot 139) \\
\]
\[ a_1 = 2^3 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j \cdot (1847032397279 \cdot j^{11} - 2^6 \cdot 47 \cdot 157 \cdot 3691 \cdot 11660843 \cdot j^{10} \\
+ 2^{14} \cdot 3^4 \cdot 383 \cdot 25679 \cdot 7797631 \cdot j^9 - 2^{20} \cdot 3^6 \cdot 400129001343469 \cdot j^8 \\
+ 2^{24} \cdot 3^9 \cdot 5 \cdot 41 \cdot 503 \cdot 67307 \cdot 267271 \cdot j^7 \\
- 2^{30} \cdot 3^{12} \cdot 19 \cdot 509 \cdot 13597 \cdot 11431571 \cdot j^6 \\
+ 2^{37} \cdot 3^{15} \cdot 31 \cdot 3038701 \cdot 4610147 \cdot j^5 - 2^{43} \cdot 3^{20} \cdot 7^2 \cdot 41 \cdot 73 \cdot 2381 \cdot 56891 \cdot j^4 \\
+ 2^{52} \cdot 3^{21} \cdot 5 \cdot 139 \cdot 9239401667 \cdot j^3 - 2^{62} \cdot 3^{25} \cdot 5 \cdot 1381 \cdot 3698087 \cdot j^2 \\
+ 2^{73} \cdot 3^{29} \cdot 11 \cdot 47 \cdot 58693 \cdot j - 2^{85} \cdot 3^{33} \cdot 8161) \\
\]
\[ a_0 = -2^4 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j \cdot (2^3 \cdot 3^2 \cdot 7^6 \cdot j^{14} - 5 \cdot 13 \cdot 3109 \cdot 76441597 \cdot j^{13} \\
+ 2^4 \cdot 3449 \cdot 4363 \cdot 873750089 \cdot j^{12} - 2^{11} \cdot 3^4 \cdot 7 \cdot 2087 \cdot 57859 \cdot 9420337 \cdot j^{11} \\
+ 2^{16} \cdot 3^8 \cdot 11^2 \cdot 73 \cdot 125183 \cdot 10636957 \cdot j^{10} - 2^{26} \cdot 3^9 \cdot 691 \cdot 14434308694753 \cdot j^9 \\
+ 2^{31} \cdot 3^{13} \cdot 101 \cdot 283 \cdot 252059913139 \cdot j^8 \\
- 2^{37} \cdot 3^{16} \cdot 11 \cdot 13 \cdot 17 \cdot 647 \cdot 863 \cdot 4253233 \cdot j^7 \\
+ 2^{43} \cdot 3^{18} \cdot 631819 \cdot 1645187913 \cdot j^6 - 2^{48} \cdot 3^{23} \cdot 149 \cdot 233 \cdot 90533 \cdot 330413 \cdot j^5 \\
+ 2^{59} \cdot 3^{25} \cdot 23 \cdot 1408302006413 \cdot j^4 - 2^{70} \cdot 3^{27} \cdot 726838208711 \cdot j^3 \\
+ 2^{80} \cdot 3^{32} \cdot 7 \cdot 263 \cdot 337 \cdot 1327 \cdot j^2 - 2^{90} \cdot 3^{37} \cdot 569731 \cdot j + 2^{100} \cdot 3^{39} \cdot 173) \\
\]
\[ b_{11} = -(j - 2^6 \cdot 3^3) \cdot j \\
\]
\[ b_{10} = -2 \cdot 13 \cdot 3^2 \cdot (j - 2^6 \cdot 3^3) \cdot j^2 \\
\]
\[ b_9 = 2^2 \cdot (j - 2^3 \cdot 3^6) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \\
\]
\[ b_8 = 3^4 \cdot (13 \cdot j - 2^5 \cdot 3 \cdot 163) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \\
\]
\[b_7 = 5 \cdot 2^5 \cdot 3^6 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4\]
\[b_6 = 2^2 \cdot 3^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (j^2 + 2^4 \cdot 3^5 \cdot 13 \cdot j - 2^9 \cdot 3^5 \cdot 269)\]
\[b_5 = 2^5 \cdot 3^5 \cdot (5 \cdot j - 2^6 \cdot 3^4) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5\]
\[b_4 = 2^8 \cdot 3^8 \cdot (31 \cdot j - 2^3 \cdot 3^2 \cdot 1471) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^6\]
\[b_3 = 2^8 \cdot 3^8 \cdot (383 \cdot j - 2^6 \cdot 3 \cdot 1447) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^6\]
\[b_2 = 2^9 \cdot 3^9 \cdot (3923 \cdot j - 2^6 \cdot 3^5 \cdot 139) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^7\]
\[b_1 = 13 \cdot 19 \cdot 3^{11} \cdot 2^{15} \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8\]
\[b_0 = -2^8 \cdot 3^9 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (j^2 - 2^7 \cdot 3^3 \cdot 1399 \cdot j + 2^{12} \cdot 3^6 \cdot 17^3)\]

References


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