AN INFINITE FAMILY OF LACUNARY SERIES

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Abstract. Inspired by Andrews’ work, we study the \( q \)-series
\[
f(a, b, c; q) = (q)\infty \sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(q)_n^2(1-q^{bn+c})}.
\]
In particular, we prove a relation between \( f(2, 1, n; q) \) and \( f(2, 1, n+1; q) \) which implies that \( f(2, 1, m; q) \) is lacunary for all \( m \geq 1 \). Using work of Coogan and Ono, we conclude with a new partition identity involving the number of partitions with positive crank and the number of partitions with no part equal to its Durfee square.

1. Introduction and Statement of Results

In [1], Andrews considers the \( q \)-series
\[
f(a, b, c; q) = (q)\infty \sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(q)_n^2(1-q^{bn+c})}
\]
where \( (q)_k = \prod_{n=1}^{k} (1-q^n) \), and utilizes specific instances of it to describe the generating functions for the number of concave compositions of certain forms. In the course of [1], the following identities relating to \( f \) arise:
\[
\begin{align*}
f(1, 2, 2; q) &= \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} (1-q^{2n+1}) \\
f(3, 2, 1; q) &= \sum_{n=1}^{\infty} \left(-q^{6n^2-2n-1} + q^{6n^2+2n-1} - (-1)^n q^{n(n+1)/2} - 1\right) \\
f(2, 1, 1; q) &= \sum_{n=0}^{\infty} (-1)^n q^{n^2+3n}.
\end{align*}
\]

A \( q \)-series \( \sum a_n q^n \) is called lacunary if
\[
\lim_{X \to \infty} \frac{\#\left\{ n \leq X : a_n \neq 0 \right\}}{X} = 0.
\]

It is clear that the three series above are lacunary. We expect lacunary \( f(a, b, c; q) \) to be quite rare. In this paper we present an infinite family of lacunary series of the form \( f(a, b, c; q) \) and a recurrence-type relation between them. To be precise, we have the following theorem.

Theorem 1.1. Let \( m \in \mathbb{Z}^+ \). Then \( f(2, 1, m; q) - qf(2, 1, m+1; q) \) is a polynomial in \( q \). Moreover, \( f(2, 1, m; q) \) is lacunary and we have that
\[
\lim_{m \to \infty} (f(2, 1, m; q) - qf(2, 1, m+1; q)) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2+3n}.
\]
While there are other families that exhibit similar properties, this family is particularly interesting because of a partition-theoretic result that arises from these results. If a partition \( a_k + \cdots + a_0 \) of \( n \) has \( r \) ones and \( s \) the number of parts of that partition larger than \( r \), then that crank of that partition is

\[
\begin{cases} 
    a_k & \text{if } r = 0 \\
    s - r & \text{if } r > 0
\end{cases}
\]

The crank first arose in combinatorial proofs of Ramanujan’s congruences (see [2], [8], and [3].) Let \( p(n) \) denote the number of partitions of \( n \), \( u(n) \) denote the number of partitions of \( n \) with positive crank, and \( v(n) \) the number of partitions of \( n \) with no parts equal to the size of their Durfee square. By studying the series above, we are led to the following partition identity.

**Theorem 1.2.** For all \( n \), we have that

\[
p(n) = u(n + 1) + u(n) + v(n + 1) - v(n).
\]

**Example.** The seven partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1. Out of these, only 5, 3+2, and 2+2+1 have positive crank. The only partitions of 6 with positive crank are 6, 4+2, 3+3, 3+2+1, and 2+2+2. The only partition of 5 with no part equal to its Durfee square is 5. The only partitions of 6 with no part equal to its Durfee square are 6 and 3+3. Hence \( p(5) = 7, u(6) = 5, u(5) = 3, v(5) = 1 \) and \( v(6) = 2 \)

\[
p(5) = u(6) + u(5) + v(5) - v(6).
\]

2. **Results**

In order to prove the main theorem, we will first need two lemmas to deal with series that look similar to \( f(2, 1, c; q) \). If \( k \in \mathbb{Z}^+ \cup 0 \), then from [4] we have the following identity:

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+kn}}{(q^n)^2(1-q^{n+1})\ldots(1-q^{n+k})} = \frac{1}{(q)_\infty}.
\]

This result will be essential in the proofs of the lemmas in this section.

**Lemma 2.1.** If \( m \geq 1 \), then

\[
p_m := (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+kn}}{(q^n)^2(1-q^{n+1})\ldots(1-q^{n+k})\ldots(1-q^{n+k+m})}
\]

is a polynomial in \( q \).
Proof. We note that

\[ p_m = (q) \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+1}) \ldots (1 - q^{n+k+m})} \]

\[ = (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}(1 - q^{n+m+1})}{(q)_n^2(1 - q^{n+1}) \ldots (1 - q^{n+m})(1 - q^{n+k+m+1})} \]

\[ = (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+1}) \ldots (1 - q^{n+m})(1 - q^{n+k+m+1})} \]

so the lemma follows by induction on \( m \) along with 2.1.

\[ \blacksquare \]

**Lemma 2.2.** If \( m, k \in \mathbb{Z}^+ \geq 1 \), then

\[ p_{m,k} := (q) \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+m}) \ldots (1 - q^{n+m+k-1})} \]

is a polynomial in \( q \).

Proof. We have that

\[ p_{m,k} = (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+m}) \ldots (1 - q^{n+m+k-1})} \]

\[ = (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}(1 - q^{n+m-1})}{(q)_n^2(1 - q^{n+m-1})(1 - q^{n+m}) \ldots (1 - q^{n+m+k-1})} \]

\[ = (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+m-1})(1 - q^{n+m}) \ldots (1 - q^{n+m+k-1})} \]

\[ - (q)^{m+k} \sum_{n=0}^{\infty} \frac{q^{n^2 + kn}}{(q)_n^2(1 - q^{n+m-1})(1 - q^{n+m}) \ldots (1 - q^{n+m+k-1})} \]

and the lemma follows by induction on \( m \) and Lemma 2.1.

\[ \blacksquare \]

We are now prepared to prove the main theorem of this paper. For ease of notation, define

\[ P_n(q) := f(2, 1, m; q) - qf(2, 1, m + 1; q) \]

and

\[ P_\infty(q) := \lim_{n \to \infty} P_n(q). \]

**Proof of Theorem 1.1.** Let \( m \in \mathbb{Z}^+ \). Then we have

\[ f(2, 1, m; q) - qf(2, 1, m + 1; q) = (q)^{m+1} \sum_{n=0}^{\infty} \left( \frac{q^{n^2 + 2n}}{(q)_n^2(1 - q^{n+m})} + \frac{q^{n^2 + 2n+1}}{(q)_n^2(1 - q^{n+m+1})} \right) \]

\[ = (1 - q) (q)^{m+1} \sum_{n=0}^{\infty} \frac{q^{n^2 + 2n}}{(q)_n^2(1 - q^{n+m})(1 - q^{n+m+1})}, \]
which is a polynomial by Lemma 2.2. Since \( f(2, 1, 1; q) \) is lacunary, it follows immediately that \( f(2, 1, 1; m) \) is lacunary.

Now, we clearly have

\[
P_\infty(q) = (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q)_n^2}.
\]

By Example 23 of Chu and Zhang [6], the right hand side of this equation is equal to

\[
1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}}.
\]

Recall that \( p(n) \) is the number of partitions of \( n \), \( u(n) \) is the number of partitions of \( n \) with positive crank, and is \( v(n) \) the number of partitions of \( n \) with no parts equal to the size of their Durfee square. From [5], we have that

\[
\sum_{n=0}^{\infty} u(n+1)q^n = \frac{1}{(q)_\infty} f(2, 1, 1; q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q)_n^2 (1 - q^{n+1})}
\]

Similarly, we have that

\[
\sum_{n=0}^{\infty} v(n+2)q^n = \frac{1}{(q)_\infty} f(3, 1, 1; q)
= \frac{1}{q^2} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2+(n+1)}}{(q)_n^2 (1 - q^{n+1})}
\]

by noting that the \( q^{(n+1)^2+(n+1)} \) term dictates the size of the Durfee square of a partition and the \( (q)_n^2 \) and \( (1 - q^{n+1}) \) terms ensure that only the partitions wanted are counted.

**Proof of Theorem 1.2.** In [7], Coogan and Ono prove that \( 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} \) satisfies the following identity:

\[
1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} = 1 - 2q f(2, 1, 1; q).
\]

By Theorem 1.1, this implies that

\[
(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q)_n^2} = 1 - 2(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n+1}}{(q)_n^2 (1 - q^{n+1})}.
\]

This yields

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+2n} + q^{n^2+2n+1} + q^{n^2+3n+2} - q^{n^2+3n+1}}{(q)_n^2 (1 - q^{n+1})} = \frac{1}{(q)_\infty},
\]

which is equivalent to

\[
\frac{1}{(q)_\infty} \left( f(2, 1, 1; q) + q f(2, 1, 1; q) + q^2 f(3, 1, 1; q) - q f(3, 1, 1; q) \right) = \frac{1}{(q)_\infty},
\]
Hence, since $\frac{1}{(q)_{\infty}}$ is the generating function for $p(n)$ and $\frac{1}{(q)_{\infty}} f(2, 1, 1; q)$ and $\frac{1}{(q)_{\infty}} f(3, 1, 1; q)$ are the generating functions for $u(n + 1)$ and $v(n + 2)$, respectively,

$$p(n) = u(n + 1) + u(n) + v(n + 1) - v(n).$$

\[\square\]

References


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