POLYNOMIALS THAT BEHAVE LIKE THE RIEMANN ZETA-FUNCTION

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Abstract. In this paper, revisiting work of Rodriguez-Villegas [3], we produce infinite families of polynomials that satisfy the essential expected properties of the Riemann zeta-function. We identify natural families of rational functions in x which are the generating functions for the values of "zeta-polynomials" $Z_T(s)$. In analogy with the zeta-function, these polynomials satisfy a functional equation of the form

$$Z_T(s) = (-1)^t Z_T(1-s),$$

and enjoy the additional property that if $Z_T(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$. Namely, these polynomials satisfy the Riemann Hypothesis.

1. Introduction and Statement of Results

The Riemann zeta-function $\zeta(s)$ (for background information, see [1]) is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

where s is a complex number with $\mathrm{Re}(s) > 1$. This function has a pole at $s = 1$, which is immediately apparent by the divergence of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This guarantees the infinitude of primes, because otherwise the zeta-function would not diverge at $s = 1$, as it would become a non-vanishing finite product. This function is central to the study of the distribution of primes.

The Riemann zeta-function also admits an analytic continuation (apart from the pole at $s = 1$) to the complex plane. Let

$$\Lambda(s) := \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be the completed Riemann zeta-function, where $\Gamma(s)$ is the Gamma-function as defined by the integral

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-s} ds.$$

The Lambda-function has the property that

$$\Lambda(s) = \Lambda(1-s),$$

known as a functional equation. By making use of the functional equation and the properties of the Gamma-function, we arrive at the analytic continuation of $\zeta(s)$ to the entire complex plane. Note that the complex Gamma-function has poles at all non-positive integers. When s is a negative even integer, $\frac{s}{2}$ would be a negative integer and would thus be a pole of $\Gamma(\frac{s}{2})$. Therefore, since $\Lambda(s)$ is well-defined at these values by the functional equation, all negative even integers are zeros for $\zeta(s)$, which we call *trivial zeros*. All other zeros of the zeta-function are known as *non-trivial zeros*. Although much is known about the Riemann zeta-function, little is known about its non-trivial zeros.

Conjecture (Riemann Hypothesis)**.** *If $\zeta(\rho) = 0$ and ρ is a non-trivial zero, then* $\mathrm{Re}(\rho) = \frac{1}{2}$.

A natural question is whether there are infinite families of polynomials $Z(s)$ that emulate the Riemann zeta-function, which we call *zeta-polynomials*. To be a zeta-polynomial, we require that $Z(s) = \pm Z(1-s)$ and that if $Z(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$. Expanding on work of Rodriguez-Villegas [3], we answer this question by considering rational functions that generate zeta-polynomials and determining the zeros of the generating functions of these rational functions.

Define the set

$$T := \{0 < \theta_j < 2 : 1 \leq j \leq t\}.$$

Define a rational function $F_T(x)$ by

$$F_T(x) := \frac{\prod_{j=1}^t (x - e^{\pi i \theta_j})}{(x-1)^{t+1}} = \frac{P_T(x)}{(x-1)^{t+1}}, \tag{1.1}$$

where all the roots of $F_T(x)$ lie on the unit circle.

We then define $Z_T(-s)$ as a polynomial that is the coefficient of x^n generated from the power series of these rational functions $F_T(x)$ in the following theorem.

Theorem 1.1. *Assuming the notation above, there is a polynomial $Z_T(s)$ of degree t such that*

$$F_T(x) = \sum_{n \geq 0} Z_T(-n) x^n.$$

Furthermore, if $Z_T(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$.

Remark 1. Theorem 1.1 includes the specific property that the degree of the numerator of rational function $F_T(x)$ is one less than the degree of the denominator. It turns out that in order to prove Theorem 1.1, we need to understand the general case when the degree of the numerator is smaller than the degree of the denominator in the rational function. Doing so, we re-prove the main theorem in [3] through similar but different methods. In particular, our proof makes explicit how to obtain the polynomial $Z_T(X)$ from a given set T.

Remark 2. If we relaxed Theorem 1.1 to include roots of the rational function that do not lie on the unit circle, then not all roots ρ will have the property $\mathrm{Re}(\rho) \neq \frac{1}{2}$ when $Z_T(\rho) = 0$.

Theorem 1.1 gives infinite families of polynomials that arise from rational functions whose roots lie on the unit circle. Given that the real part of the roots of $Z_T(s)$ is $\frac{1}{2}$, it is natural to ask when $Z_T(s)$ satisfies a functional equation of an analogous form to the zeta-function's functional equation.

Theorem 1.2. *Assume the notation above. If $Z_T(s) \in \mathbb{R}[s]$, then*

$$Z_T(s) = (-1)^t Z_T(1-s).$$

Remark 3. If $P_T(x) \in \mathbb{R}[x]$, then $Z_T(s) \in \mathbb{R}[s]$.

Remark 4. Theorem 1.2 is a special case of the corollary in [3]. We offer an original proof.

To demonstrate the utility of Theorem 1.2, we offer a corollary that provides an infinite family of polynomials that satisfy the properties of the Riemann Hypothesis. We can use the equivalence

$$\binom{n}{k} = \frac{\prod_{j=0}^{k-1}(n-j)}{k!}$$

to show that the following binomial is a polynomial in s.

Corollary 1.3. *Let $T = \{\frac{1}{m}, \frac{2}{m}, \cdots, \frac{2m-1}{m}\}$. The polynomial $Z_T(s)$ is*

$$Z_T(s) = (-1)^{m+1} \sum_{j=0}^{m-1} \binom{-s+j}{m-1}.$$

If $Z_T(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$. Moreover,

$$Z_T(s) = -Z_T(1-s).$$

It turns out that Corollary 1.3 is a specific case of the following corollary, which involves *cyclotomic polynomials* $\Phi_n(x)$. These are defined as [2]

$$\Phi_n(x) = \prod_{\substack{1\leq j\leq n \\ \gcd(j,n)=1}} (x - \zeta_n^j),$$

where $n \in \mathbb{Z}^+$. Note that $\deg(\Phi_n(x)) = \phi(n)$, where the function $\phi(n)$ is Euler's totient function. We can choose the $P_T(x)$ in equation (1.1) to be a finite product of an arbitrary number of cyclotomic polynomials to satisfy the properties of the zeta-function.

Corollary 1.4. *Let $m_1, \ldots, m_k$ be positive integers such that $m_j \geq 2$ and suppose T is such that*

$$P_T(x) = \prod_{j=1}^{k} \Phi_{m_j}(x).$$

If $Z_T(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$. Furthermore, $Z_T(s) = (-1)^{\sum_{j=1}^{k}\phi(m_j)} Z_T(1-s)$, where $\phi(n)$ is Euler's totient function.

Remark 5. It is straightforward to calculate the sign of $Z_T(1-s)$ in the functional equation by noting that the value of $\phi(n)$ is odd only when $n = 2$. Therefore, if none of the m_j terms equal 2, then $Z_T(s) = Z_T(1-s)$.

Example. When $m = 3$ in Corollary 1.4, the function $F_T(x)$ is

$$F_T(x) = \frac{x^2 + x + 1}{(x-1)^3} = \frac{(x - e^{\frac{2}{3}\pi i})(x - e^{\frac{4}{3}\pi i})}{(x-1)^3}.$$

The zeta-polynomial $Z_T(s)$ is

$$Z_T(s) = -3s^2 + 3s - 2,$$

and $Z_T(1-s)$ is also

$$Z_T(1-s) = -3s^2 + 3s - 2,$$

so $Z_T(s) = Z_T(1-s)$. If $Z_T(\rho) = 0$, the roots are

$$\rho \approx 0.5 \pm 0.6455i.$$

The constructions in the paper are based on the coefficients of the reciprocal of the powers of $x - 1$ forming polynomials that result from the binomial theorem. The zeros to these polynomials are easily identifiable. Upon increasing the degree of the numerator of the rational function, other zeros emerge. We focus on locating the new zeros generated by this process and determine the functional equation satisfied by the polynomial.

The paper is organized as follows. In Section 2, we introduce the lemmas needed to prove that the zeros of $Z_T(s)$ have real part $\frac{1}{2}$ and determine the functional equation of $Z_T(s)$. In Section 3, we prove Theorem 1.1 and Theorem 1.2. In Section 4, we illustrate numerical examples of zeta-polynomials.

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2. Preliminaries

To prove Theorem 1.1 and Theorem 1.2, we study the coefficients of the power series expansions for rational functions $F_T(x)$, as defined in Theorem 1.1. It turns out that the coefficients are special values of polynomials, which we refer to as *associated power series polynomials*. In Section 2.1, we construct lemmas that determine the properties of the zeros of associated power series polynomials. In Section 2.2, we use the results from Section 2.1 to build functional equations for suitable power series polynomials.

2.1. **Zeros of Zeta-Polynomials.** Our proof of Theorem 1.1 relies on understanding zeros of associated power series polynomials, which are studied in Lemmas 2.1-2.4.

We first find the formal power series for the rational function

$$G_m(x) = \frac{1}{(x-1)^m}. \tag{2.1}$$

We make use of the fact that a binomial containing the variable s is a polynomial in s due to the equivalence

$$\binom{n}{k} = \frac{\prod_{j=0}^{k-1}(n-j)}{k!}.$$

Lemma 2.1. *Let $m \in \mathbb{Z}^+$ and let*

$$W_m(s) = (-1)^{m+1}\binom{s+m-1}{m-1} = \left(\frac{1}{(m-1)!}\left(\prod_{j=1}^{m-1}(s+j)\right)\right)(-1)^{m+1}.$$

Then we have

$$G_m(x) := \frac{1}{(x-1)^m} = \sum_{n\geq 0} W_m(-n)x^n,$$

Proof. This can be proven by induction on m by differentiating the power series. Our base case is when $m = 1$, which yields

$$\frac{1}{x-1} = -\sum_{n\geq 0} x^n$$

by the classical geometric series. Differentiating both sides and simplifying, we get

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x-1}\right] &= \frac{d}{dx}\left[-\sum_{n\geq 0} x^n\right]\\ \frac{-1}{(x-1)^2} &= -\sum_{n\geq 0} nx^{n-1}\\ \frac{-1}{(x-1)^2} &= -\sum_{n\geq 1} nx^{n-1},\end{aligned}$$

since when $n = 0$, the first term of the sum is 0. Shifting the exponent of x back to n results in

$$\begin{aligned}\frac{-1}{(x-1)^2} &= -\sum_{n\geq 0}(n+1)x^n\\ \frac{1}{(x-1)^2} &= \sum_{n\geq 0}(n+1)x^n.\end{aligned}$$

The base case is found to be true. Suppose the equation in the lemma is true for $m \leq k$ where $k \in \mathbb{Z}^+$. Then

$$\frac{1}{(x-1)^k} = (-1)^{k+1}\sum_{n\geq 0}\frac{1}{(k-1)!}\left(\prod_{j=1}^{k-1}(n+j)\right)x^n. \tag{2.2}$$

Differentiating both sides of equation (2.2) results in

$$\frac{d}{dx}\left[\frac{1}{(x-1)^k}\right] = \frac{d}{dx}\left[(-1)^{k+1}\sum_{n\geq 0}\frac{1}{(k-1)!}\left(\prod_{j=1}^{k-1}(n+j)\right)x^n\right]. \tag{2.3}$$

After simplifying and shifting the summation index, the equation above becomes

$$\frac{1}{(x-1)^{k+1}} = (-1)^{k+2}\sum_{n\geq 0}\frac{1}{k!}\Big(\prod_{j=1}^{k}(n+j)\Big)x^n, \tag{2.4}$$

which is our formula but with k replaced with $k+1$. This means that the inductive step holds, so for all $m \in \mathbb{Z}^+$, the equation in the lemma is true. $\square$

Note that the function $W_m(-n)$ has zeros at $-1, -2, \cdots, -(m-1)$. In analogy with the zeta-function, we call these zeros of $W_m(n)$ *trivial zeros*. Notice that the previous lemma demonstrated that these zeros result from factorials, which is reminiscent of the Gamma-function and how it creates trivial zeros for the Riemann zeta-function.

The next lemma shows that one of the trivial zeros is eliminated and a complex root appears when $G_m(x)$ is multiplied by the factor $x - e^{\pi i\theta}$.

Lemma 2.2. *If we let*

$$\sum_{n\geq 0} Y_1(-n)x^n := G_m(x)(x - e^{\pi i\theta}),$$

then $Y_1(-s)$ has trivial zeros at $-1, -2, \cdots, -(m-2)$ and a complex root with real part $-\frac{m-1}{2}$.

Proof. Let $H_1(x) = (x - e^{\pi i\theta})G_m(x)$. Then, we can express the associated power series polynomial of $H_1(x)$ using $W_m(n) = \frac{1}{(m-1)!}\prod_{j=1}^{m-1}(n+j)$, the power series polynomial of $G_m(x)$, as

$$G_m(x) = -\sum_{n\geq 0} \frac{1}{(m-1)!}\left(\prod_{j=1}^{m-1}(n+j)\right)x^n.$$

Multiplying by $x - e^{\pi i\theta}$, we get

$$\begin{aligned}(x - e^{\pi i\theta})G_m(x) &= -\left(\sum_{n\geq 1} \frac{1}{(m-1)!}\left(\prod_{j=0}^{m-2}(n+j)\right)x^{n+1}\right) \\ &\quad + e^{\pi i\theta}\left(\sum_{n\geq 0} \frac{1}{(m-1)!}\left(\prod_{j=1}^{m-1}(n+j)\right)x^n\right) \\ (x - e^{\pi i\theta})G_m(x) &= \left(-\frac{1}{m-1!}\prod_{j=1}^{m-2}(n+j)\right)\bigl(n - e^{\pi i\theta}(n+m-1)\bigr).\end{aligned}$$

This means that $H_0(x)$ has $m-2$ trivial zeros and one complex root. Define $g_1(n)$ as

$$g_1(n) = \bigl(n - e^{\pi i\theta}(n+m-1)\bigr).$$

If $g_1(n) = 0$, then

$$n - e^{\pi i\theta}(n+m-1) = 0.$$

Solving for n in the above equation returns

$$n = \frac{(m-1)e^{\pi i\theta}}{1 - e^{\pi i\theta}}.$$

Then, we multiply both the numerator and denominator by the complex conjugate of $1-e^{\pi i\theta}$, which is $1-e^{-\pi i\theta}$, to get

$$n=\frac{(m-1)e^{\pi i\theta}}{1-e^{\pi i\theta}}\cdot\frac{1-e^{-\pi i\theta}}{1-e^{-\pi i\theta}}=\frac{(m-1)e^{\pi i\theta}-m+1}{2-e^{\pi i\theta}-e^{-\pi i\theta}}.$$

The substitution of trigonometric identities simplifies n to be

$$\begin{aligned}n&=\frac{(m-1)\cos(\pi\theta)+(m-1)i\sin(\pi\theta)-m}{2-2\cos(\pi\theta)}\\&=\frac{\frac{m-1}{2}\cos(\pi\theta)+\frac{m-1}{2}i\sin(\pi\theta)-\frac{m-1}{2}}{1-\cos(\pi\theta)}\\&=\frac{-\frac{m-1}{2}(1-\cos(\pi\theta)}{1-\cos(\pi\theta)}+\frac{\frac{m-1}{2}\sin(\pi\theta)}{1-\cos(\pi\theta)}i.\end{aligned}$$

This reduces to

$$n=-\frac{m-1}{2}+\frac{\frac{m-1}{2}\sin(\pi\theta)}{1-\cos(\pi\theta)}i,$$

which shows that when we multiply $G_m(x)$ by $x-e^{\pi i\theta}$, a complex root of $Y_1(n)$ emerges and has real part $-\frac{m-1}{2}$. □

Lemma 2.2 is generalized in the following two lemmas, where we multiply $G_m(x)$ by more $x-e^{\pi i\theta_j}$ factors. Furthermore, H_1 and Y_1 from the previous lemma are extended to H_k and Y_k, where k is the number of $x-e^{\pi i\theta_j}$ factors in the numerator of the rational function.

Lemma 2.3. *Let the set T be defined as*

$$T:=\{0<\theta_j<2:1\le j\le t\}.$$

Suppose $m=t+1$. For $1\le k<m$, let

$$H_k(x):=\frac{\prod_{j=1}^{k}(x-e^{\pi i\theta_j})}{(x-1)^m}.$$

Then there is a polynomial $Y_k(s)$ such that $H_k(x)=\sum_{n\ge0}Y_k(-n)x^n$. The trivial zeros of $Y_k(-s)$ are $-1,-2,\dots,-(m-k-1)$. Moreover, $Y_k(-s)$ factors as

$$Y_k(-s)=(-1)^{m+1}\frac{1}{(m-1)!}\left(\prod_{\ell=-(m-k-1)}^{-1}(s-\ell)\right)\cdot g_k(s),$$

where $g_k(s)$ is a polynomial of degree k and the relationship between $g_k(n)$ and $g_{k-1}(n)$ is

$$g_k(n):=-e^{\pi i\theta_k}(n+m-k)g_{k-1}(n)+g_{k-1}(n-1)n.$$

Proof. We can prove this lemma by inducting on k. By Lemma 2.2, we know that our base case is true. To begin our inductive step, we assume the lemma is true for some $k < m$. Then we have that

$$\frac{\prod_{j=1}^{k}(x-e^{\pi i\theta_j})}{(x-1)^m} = (-1)^{m+1}\frac{1}{(m-1)!}\sum_{n\geq 0}\left(\prod_{\ell=-(m-k-1)}^{-1}(n-\ell)\right)\cdot g_k(n)x^n.$$

Multiplying by $x - e^{\pi i\theta_{k+1}}$, we obtain

$$\begin{aligned}\frac{\prod_{j=1}^{k+1}(x-e^{\pi i\theta_j})}{(x-1)^m} = (-1)^{m+1}\frac{1}{(m-1)!}\Bigg(&\sum_{n\geq 0}\left(\prod_{\ell=-(m-k-1)}^{-1}(n-\ell)\right)\cdot g_k(n)x^{n+1}\\ &- e^{\pi i\theta_{k+1}}\sum_{n\geq 0}\left(\prod_{\ell=-(m-k-1)}^{-1}(n-\ell)\right)\cdot g_k(n)x^n\Bigg),\end{aligned}$$

which simplifies to

$$\begin{aligned}\frac{\prod_{j=1}^{k+1}(x-e^{\pi i\theta_j})}{(x-1)^m} = (-1)^{m+1}\frac{1}{(m-1)!}\Bigg(&\sum_{n\geq 1}\left(\prod_{\ell=-(m-k-2)}^{-2}(n-\ell)\right)\cdot g_k(n-1)x^n\\ &- e^{\pi i\theta_{k+1}}\sum_{n\geq 0}\left(\prod_{\ell=-(m-k-1)}^{-1}(n-\ell)\right)\cdot g_k(n)x^n\Bigg).\end{aligned}$$

Again, we can factor out $(-1)^{m+1}\frac{1}{(m-1)!}\prod_{\ell=-(m-k-1)}^{-2}(n-\ell)$ to get

$$\begin{aligned}\frac{\prod_{j=1}^{k+1}(x-e^{\pi i\theta_j})}{(x-1)^m} = &\left((-1)^{m+1}\frac{1}{(m-1)!}\prod_{\ell=-(m-k-1)}^{-2}(n-\ell)\right)\\ &\cdot\left(\sum_{n\geq 1}(n-m+k+2)\cdot g_k(n-1)x^n - e^{\pi i\theta_{k+1}}\sum_{n\geq 0}(n+1)\times g_k(n)x^n\right).\end{aligned}$$

Then $Y_{k+1}(n)$ equals

$$\begin{aligned}Y_{k+1}(n) = (-1)^{m+1}\frac{1}{(m-1)!}&\left(\prod_{\ell=-(m-k-1)}^{-2}(n-\ell)\right)((n-m+k+2)g_k(n-1)\\ &- e^{\pi i\theta_{k+1}}(n+1)g_k(n)).\end{aligned}$$

Therefore, $Y_{k+1}(n)$ simplifies to

$$Y_{k+1}(n) = (-1)^{m+1}\frac{1}{(m-1)!}\left(\prod_{\ell=-(m-k-1)}^{-2}(n-\ell)\right)(g_{k+1}(n)),$$

as desired. The inductive step is now complete. □

The following lemma builds on the previous lemma to show that $\mathrm{Re}(\rho) = -\frac{m-k}{2}$ when $Y_k(-\rho) = 0$.

Lemma 2.4. *Assuming the notation above, if* $1 < k < t$ *and* $g_k(\rho) = 0$*, then* $\mathrm{Re}(\rho) = -\frac{m-k}{2}$.

Remark 6. This lemma, together with Lemma 2.3, should be compared with the lemma in [3], which provides a similar inductive step.

Remark 7. Let $m-k=1$. If $Y_k(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$.

Proof. The base case is found true by Lemma 2.2. Suppose that if $g_{k-1}(\rho) = 0$, then $\mathrm{Re}(\rho) = -\frac{m-(k-1)}{2}$.

To find the roots ρ to $g_k(\rho) = 0$, we set the recursive form of $g_k(\rho)$ to 0:

$$g_k(\rho) = -e^{\pi i\theta_k}(\rho+m-k)g_{k-1}(\rho) + g_{k-1}(\rho-1)\rho = 0,$$

which we found in Lemma 2.3. Moving the left term to the righthand side and taking the norms on both sides of the equation, we get

$$|g_{k-1}(\rho-1)||\rho| = |e^{\pi i\theta_k}||\rho+m-k||g_{k-1}(\rho)|. \tag{2.5}$$

By the inductive hypothesis, we can factor the function $g_{k-1}(\rho) = \prod_{j=1}^{k-1}(\rho-\alpha_j)$, where $\alpha_j = -\frac{m-k+1}{2} + c_j i$ for real number c_j. Substituting $\rho = a+bi$, equation (2.5) can be expressed as

$$\left(\prod_{j=1}^{k-1}|\rho-1-\alpha_j|\right)|\rho| = \left(\prod_{j=1}^{k-1}|\rho-\alpha_j|\right)|\rho+m-k|. \tag{2.6}$$

After computing and squaring the norms, as well as substituting all variables, the lefthand side of equation (2.6) becomes

$$\left(\prod_{j=1}^{k-1}(a+\tfrac{m-k-1}{2})^2+(b-c_j)^2\right)(a^2+b^2). \tag{2.7}$$

Likewise, the righthand side of equation (2.6) can be expressed as

$$\left(\prod_{j=1}^{k-1}(a+\tfrac{m-k+1}{2})^2+(b-c_j)^2\right)\left((a+m-k)^2+b^2\right). \tag{2.8}$$

Let $a=-\frac{m-k}{2}+\epsilon$. To prove that $\mathrm{Re}(\rho)=-\frac{m-k}{2}$ when $g_k(\rho)=0$, we want to show that $\epsilon=0$. Define the polynomial $p(\epsilon)$ to be the righthand side of equation (2.6), so the lefthand side would be $p(-\epsilon)$.

Thus, $p(\epsilon)$ is expressed as

$$\left(\prod_{j=1}^{k-1}(\epsilon+\tfrac{1}{2})^2+(b-c_j)^2\right)\left((\epsilon+\tfrac{m-k}{2})^2+b^2\right) \tag{2.9}$$

and $p(-\epsilon)$ is expressed as

$$\left(\prod_{j=1}^{k-1}(\epsilon-\tfrac{1}{2})^2+(b-c_j)^2\right)\left((\epsilon-\tfrac{m-k}{2})^2+b^2\right). \tag{2.10}$$

Let $d_j=b-c_j$ and $r=\frac{m-k}{2}$, then substitute in d_j and r into $p(\epsilon)$ and $p(-\epsilon)$ in expressions (2.9) and (2.10). This results in the terms of $p(\epsilon)$ all having positive coefficients. For $p(-\epsilon)$, the odd-degree terms have negative coefficients. Let

$$q(\epsilon):=p(\epsilon)-p(-\epsilon),$$

so $q(\epsilon)$ has all the odd-degree terms with positive coefficients. Note that ϵ divides $q(\epsilon)$. Then, we can factor $q(\epsilon)$ as

$$q(\epsilon)=\epsilon\cdot\frac{q(\epsilon)}{\epsilon}.$$

Since $q(\epsilon)$ has positive coefficients and each term in $\frac{q(\epsilon)}{\epsilon}$ has an even power, $\frac{q(\epsilon)}{\epsilon}$ is nonzero. Thus the only solution to $q(\epsilon)=0$ is when $\epsilon=0$. Therefore, $\mathrm{Re}(\rho)=a=-\frac{m-k}{2}$ when $g_k(\rho)=0$, as desired and the proof by induction is complete. □

2.2. **Functional Equation.** Our proof of Theorem 1.2 requires us to determine the functional equation for the zeta-polynomial $Z_T(s)$. We start with a more general observation about polynomials, all of whose zeros have real part $\frac{1}{2}$.

Lemma 2.5. *Let $X(s)$ be a polynomial of degree d such that $X(s)\in\mathbb{R}[s]$ such that if $X(\rho)=0$, then $\mathrm{Re}(\rho)=\frac{1}{2}$. Then we have that*

$$X(s)=(-1)^dX(1-s).$$

Remark 8. In fact, the functional equation can only hold if $X(s)\in\mathbb{R}[s]$, as is apparent in the proof of the lemma.

Proof. We will first prove that when $d \equiv 0 \pmod 2$, $X(s) = X(1-s)$. Then, we will use that to further prove that $X(s) = -X(1-s)$ when $d \equiv 1 \pmod 2$.

Let $X(s)$ be expressed as

$$X(s) = \prod_{j=1}^{d}(s - \rho_j),$$

where ρ_j represents the roots of $X(s)$. Then $X(1-s)$ can be written as

$$X(1-s) = \prod_{j=1}^{d}(1 - s - \rho_j) = \prod_{j=1}^{d} -(s - (1 - \rho_j)).$$

When d is an even integer, the negative signs from each term of $X(1-s)$ cancel out, so

$$X(1-s) = \prod_{j=1}^{d}(s - (1 - \rho_j)).$$

Since $X(s) \in \mathbb{R}[s]$, the roots of $X(s)$ form complex conjugate pairs. By assumption, we know that the real parts of the roots are equal to $\frac{1}{2}$, so the sum of each complex conjugate pair is 1. Without loss of generality, let ρ_1 and ρ_2 be a conjugate pair. This means that $\rho_1 + \rho_2 = 1$, so

$$s - \rho_1 = s - (1 - \rho_2).$$

Additionally,

$$s - \rho_2 = s - (1 - \rho_1)$$

is also true. Continuing this on for every conjugate pair, it results that

$$\prod_{j=1}^{d}(s - \rho_j) = \prod_{j=1}^{d}(s - (1 - \rho_j)),$$

which is equivalent to

$$X(s) = X(1-s).$$

This proof has shown that the functional equation must exchange a pair of roots that sum to 1. The only way for two roots with real parts $\frac{1}{2}$ to sum to 1 is when they form a conjugate pair. This implies the remark.

When d is an odd integer, we can express $X(s)$ as

$$X(s) = -(s - \rho_1)\prod_{j=2}^{d}(s - \rho_j), \tag{2.11}$$

where ρ_1 is the only root that does not form a complex conjugate pair with any other root of $X(s)$, so ρ_1 is a real number. However, by the hypotheses of the lemma, we know that ρ_1 must have real part $\frac{1}{2}$. Hence $\rho_1 = \frac{1}{2}$. Substituting in $\rho_1 = \frac{1}{2}$ and shifting the product index to start at 1, we can rewrite equation (2.11) as

$$X(s) = -\left(s - \frac{1}{2}\right)\prod_{j=1}^{d-1}(s-\rho_j). \tag{2.12}$$

Let $\widetilde{X}(s) = \prod_{j=1}^{d-1}(s-\rho_j)$. Then equation (2.12) becomes

$$X(s) = -\left(s-\frac{1}{2}\right)\widetilde{X}(s).$$

Likewise, $X(1-s)$ can be expressed as

$$X(1-s) = -\left(1-s-\frac{1}{2}\right)\widetilde{X}(1-s) = \left(s-\frac{1}{2}\right)\widetilde{X}(1-s). \tag{2.13}$$

Since $d-1$ is an even integer, we know by the first part of this proof that $\widetilde{X}(s)$ already satisfies the functional equation $\widetilde{X}(s) = \widetilde{X}(1-s)$. By use of equation (2.13),

$$\begin{aligned} X(s) &= -\left(s-\frac{1}{2}\right)\widetilde{X}(s) \\ &= -\left(s-\frac{1}{2}\right)\widetilde{X}(1-s) \\ &= -X(1-s). \end{aligned}$$

Therefore, when d is an odd integer, the functional equation for $X(s)$ is $X(s) = -X(1-s)$. This completes the proof. □

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. By Lemma 2.4, we know that $\mathrm{Re}(\rho) = -\frac{m-k}{2}$ when $g_k(\rho) = 0$. Remark 7 illustrates the specific case that when $m-k=1$, then $\mathrm{Re}(\rho) = \frac{1}{2}$ if $Y_k(\rho) = 0$. Define the set $T := \{0 < \theta_j < 2 : 1 \le j \le t\}$. Take $H_t(x)$ to be $F_T(x)$ and $Y_t(s)$ to be $Z_T(s)$ in Lemma 2.4. Then, when

$$F_T(x) = \frac{\prod_{j=1}^{t}(x - e^{\pi i \theta_j})}{(x-1)^{t+1}} = \sum_{n\ge 0} Z_T(-n)x^n,$$

we know that $\mathrm{Re}(\rho) = \frac{1}{2}$ if $Z_T(\rho) = 0$, as desired. □

Proof of Theorem 1.2. If $Z(s) \in \mathbb{R}[s]$, then by Lemma 2.5, we know that

$$Z(s) = (-1)^t Z(1-s).$$

□

4. Numerical Examples

In this section, we first demonstrate an example that illustrates only Theorem 1.1. Then, we show an example that illustrates both Theorem 1.1 and Theorem 1.2.

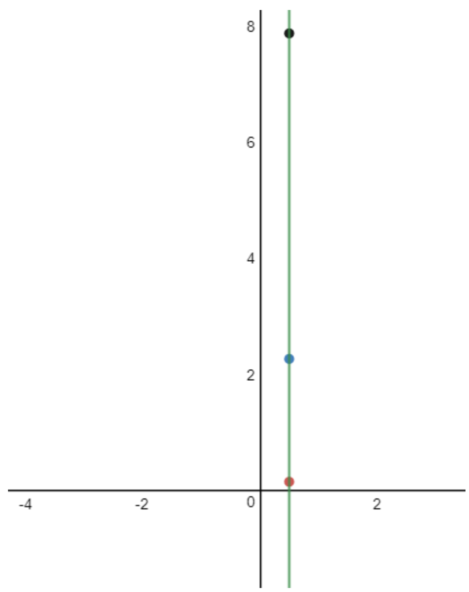
Example. Define $T := \{\frac{3}{17}, \frac{4}{17}, \frac{10}{17}\}$. Then let

$$F_T(x) := \frac{(x - e^{3i\pi/17})(x - e^{4i\pi/17})(x - e^{10i\pi/17})}{(x-1)^4}.$$

Then the zeta-polynomial $Z_T(s)$ is

$$\begin{aligned} Z_T(s) = \frac{1}{6}(6 &- (13 + e^{3i\pi/17} + e^{4i\pi/17} + e^{10i\pi/17} + 2e^{7i\pi/17} + 2e^{13i\pi/17} + 2e^{14i\pi/17})s \\ &+ 3(1 + e^{7i\pi/17} + e^{13i\pi/17} + e^{14i\pi/17})s^2 \\ &- (2 - e^{3i\pi/17} - e^{4i\pi/17} - e^{10i\pi/17} + e^{7i\pi/17} + e^{13i\pi/17} + e^{14i\pi/17})s^3). \end{aligned}$$

Consequently by Theorem 1.1, if $Z_T(\rho) = 0$, then $\mathrm{Re}(\rho) = \frac{1}{2}$. It turns out the roots are $\rho \approx 0.5 + 7.87057i, 0.5 - 0.144324i,$ and $0.5 + 2.26175i$ (graphed below).



By the remark following the statement of Lemma 2.5, it is apparent that the functional equation $Z_T(s) = -Z_T(1-s)$ does not hold.

Example. Define $T := \{\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\}$. Then the rational function $F_T(x)$ can be expanded and simplified to become

$$F_T(x) = \frac{x^6 + x^3 + 1}{(x-1)^7}.$$

Then

$$Z_T(s) = \frac{-240 - 552s - 634s^2 - 165s^3 - 85s^4 - 3s^5 - s^6}{240}.$$

If $Z_T(\rho) = 0$, the roots are $\rho \approx 0.5 \pm 8.6255i, 0.5 \pm 2.5758i,$ and $0.5 \pm 0.46583i$. Furthermore, $Z(1-s)$ is equal to

$$\frac{-240 - 552(1-s) - 634(1-s)^2 - 165(1-s)^3 - 85(1-s)^4 - 3(1-s)^5 - (1-s)^6}{240}.$$

This simplifies to

$$Z_T(1-s) = \frac{-240 - 552s - 634s^2 - 165s^3 - 85s^4 - 3s^5 - s^6}{240},$$

so the functional equation $Z(s) = Z(1-s)$ holds true.

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