Problem 0. Read Chapter II in *The foundations of mathematics* by I. Stewart and D. Tall.

Main Homework Problems

(1) Prove all of the parts of the theorem which establishes that the subsets of a set $X$ form a Boolean algebra (from class notes).

(2) Let $A = \mathbb{N} \times \mathbb{N}$. Define the relation $R$ on $A$ by

$$(m, n)R(r, s) \text{ means } m + s = r + n.$$  

Prove that $R$ is an equivalence relation.

(3) Recall the properties *reflexivity*, *symmetry*, and *transitivity* for an equivalence relation. Which of these properties is satisfied for each of the relations $R$ defined on $\mathbb{R}$ below:

- a) $x < y$
- b) $x \geq y$
- c) $|x - y| < 1$
- d) $|x - y| \leq 0$
- e) $x - y$ is rational.
- f) $x - y$ is irrational.
- g) $(x - y)^2 < 0$

(4) Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f : X \to Z$ is injective.

(5) Prove that if $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f : X \to Z$ is surjective.

(6) Let $f : X \to Y$, $\tilde{f} : X \to Y$, and $g : Y \to Z$ be functions. Prove that if $g \circ f = g \circ \tilde{f}$ and $g$ is injective, then $f = \tilde{f}$.

(7) Is the conclusion of the preceding problem true if $g$ is not injective? If not, then give an example illustrating this.

(8) Let $f : X \to Y$, $g : Y \to Z$ and $\tilde{g} : Y \to Z$ be functions. Prove that if $g \circ f = \tilde{g} \circ f$ and $f$ is surjective, then $g = \tilde{g}$.

(9) Is the conclusion of the preceding problem true if $f$ is not surjective? If not, then give an example illustrating this.

(10) Let $f : X \to Y$ be a function, and let $U$ and $V$ be subsets of $Y$. Prove that

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V).$$
(11) Determine the images of each of the functions $f : \mathbb{R} \to \mathbb{R}$ below, and say whether the function is injective, surjective, and bijective.

$$f(x) := x^3,$$
$$f(x) := x - 4,$$
$$f(x) := x^2 + \cos(x),$$
$$f(x) := |x|$$