Homework Problems

(1) Use the Euclidean Algorithm to compute \( \gcd(143, 227) \) and \( \gcd(306, 657) \).

(2) Find integers \( x \) and \( y \) for which \( \gcd(56, 72) = 56x + 72y \).

(3) Assuming that \( \gcd(a, b) = 1 \), prove the following:
   a) \( \gcd(a + b, a - b) = 1 \) or 2.
   b) \( \gcd(2a + b, a + 2b) = 1 \) or 3.

(4) Prove that if \( \gcd(a, b) = 1 \), then \( \gcd(a + b, ab) = 1 \).

(5) Which of the following equations cannot be solved in integers? Explain why.
   a) \( 6x + 51y = 22 \)
   b) \( 33x + 14y = 115 \)
   c) \( 14x + 35y = 93 \)

(6) Determine all solutions in positive integers to the following equations.
   a) \( 18x + 5y = 48 \)
   b) \( 54x + 21y = 906 \)
   c) \( 158x - 57y = 7 \)

(7) A careless bank teller mistook the number of cents for the number of dollars and vice versa when cashing a check. The customer then spends 68 cents and discovers (to his surprise) that he has twice as much money as the amount of the original check. What is the smallest value for which the check could have been written?

(8) Give an example of integers \( a, b \) and \( n \) which shows that \( a^2 \equiv b^2 \pmod{n} \) need not imply that \( a \equiv \pm b \pmod{n} \).

(9) Prove that the integer \( 53^{103} + 103^{53} \) is divisible by 39.

(10) Prove that an integer is a multiple of 9 iff the sum of its digits is a multiple of 9.

(11) For positive integers \( n \), prove that \( 7 \mid (5^{2n} + 3 \cdot 2^{5n-2}) \).
(12) Solve the following linear congruences:
   a) $25x \equiv 15 \pmod{29}$
   b) $5x \equiv 2 \pmod{26}$
   c) $36x \equiv 8 \pmod{102}$

(13) Find all the solutions to the congruence $3x - 7y \equiv 11 \pmod{13}$.

(14) Find the smallest integer $a > 2$ for which
   \[2 \mid a, \ 3 \mid (a + 1), \ 4 \mid (a + 2), \ 5 \mid (a + 3), \ 6 \mid (a + 4).\]

(15) Use Fermat’s Little Theorem to show that $17 \mid (11^{104} + 1)$.

(16) Use Fermat’s Little Theorem to show that if $n$ is a positive integer, then
   \[13 \mid (11^{12n+6} + 1).\]

(17) Suppose that $p$ is prime. If $a$ and $b$ are integers for which $a^p \equiv b^p \pmod{p}$, then prove
   that $a \equiv b \pmod{p}$.

(18) Use Fermat’s Little Theorem to prove that if $p$ is an odd prime, then both
   \[
   1^{p-1} + 2^{p-1} + 3^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p},
   1^p + 2^p + 3^p + \cdots + (p-1)^p \equiv 0 \pmod{p}.
   \]

(19) Use Wilson’s Theorem to determine the remainder obtained by dividing $15!$ by $17$.

(20) Use Wilson’s Theorem to prove (for any odd prime $p$) that
   \[
   1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.
   \]