Math 328 Homework # 3

1. Find all the solutions to the following congruences:
   a) \(7x \equiv 3 \pmod{15}\)
   b) \(6x \equiv 5 \pmod{15}\)
   c) \(x^2 \equiv 1 \pmod{8}\)
   d) \(x^2 \equiv 2 \pmod{7}\)
   e) \(x^2 \equiv 3 \pmod{7}\).

2. Find all the solutions to the following congruences:
   a) \(8x - 6 \equiv 0 \pmod{14}\)
   b) \(66x - 100 \equiv 0 \pmod{121}\)
   c) \(21x \equiv 14 \pmod{91}\).

3. Prove that 19 is not a divisor of \(4n^2 + 4\) for any integer \(n\).

4. What is the last digit in the ordinary decimal expansion of \(2^{100}\)?

5. Show that \(7 \mid (3^{2n+1} + 2^{n+2})\) for every integer \(n\).

6. Use Fermat’s Little Theorem to solve the following congruences:
   a) \(x \equiv 9^{794} \pmod{73}\)
   b) \(x^{86} \equiv 6 \pmod{29}\)
   c) \(x^{39} \equiv 3 \pmod{13}\)

7. Prove that \(n^{12} - a^{12}\) is divisible by 13 if \(a\) and \(n\) are relatively prime to 13.

8. Show that \(61! + 1 \equiv 63! + 1 \equiv 0 \pmod{71}\).

9. If \(p \equiv 3 \pmod{4}\) is prime, then show that
   \[
   \left(\frac{p-1}{2}\right)! \equiv \pm 1 \pmod{p}.
   \]