ON THE POSITIVITY OF THE NUMBER OF $t-$CORE PARTITIONS

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Abstract. A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. A Ferrers graph represents a partition in the natural way. Fix a positive integer t. A partition of n is called a $t-$*core* partition of n is none of its hook numbers are multiples of t. Let $c_t(n)$ denote the number of $t-$core partitions of n. It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$. It will be shown that if $t \geq 4$ is even, then the conjecture is true. Moreover, if $t \geq 4$, then there are at most finitely many n such that $c_t(n) = 0$. The theory of modular forms is employed here.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers with sum n. Here we define a special class of partitions.

Definition 1. Let $t \geq 1$ be a positive integer. Any partition of n whose Ferrers graph have no hook numbers divisible by t is known as a $t-$core partition of n.

The hooks are important in the representation theory of finite symmetric groups and the theory of cranks associated wiht Ramanujan's congruences for the ordinary partition function [3,4,6].

If $t \geq 1$ and $n \geq 0$, then we define $c_t(n)$ to be the number of partitions of n that are $t-$core partitions. The arithmetic of $c_t(n)$ is studied in [3,4]. The power series generating function for $c_t(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}.$$

Note that these generating functions are quotients of Dedekind $\eta-$ functions. The Dedekind $\eta-$function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

is a modular form with weight $\frac{1}{2}$. Here $q = e^{2\pi i \tau}$ is the uniformizing variable at $i\infty$.

If $p \geq 5$ is a prime, then let $\delta_p = \frac{p^2-1}{24}$. Let $a_p(n)$ be defined by:

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \sum_{n \geq \delta_p} a_p(n) q^n.$$

The arithmetic behavior of $a_p(n)$ determines the number of $p-$core partitions of n because $c_p(n-\delta_p) = a_p(n)$. Fortunately, this $\eta-$quotient is a modular form of weight $\frac{p-1}{2}$ on $\Gamma_0(p)$ with character $\epsilon(n) = \left(\frac{n}{p}\right)$. Exact formulae for $c_5(n)$ and $c_7(n)$ appear in [3].

Given a positive integer t, is $c_t(n) > 0$ for all $n \geq 0$? In other words, does every positive integer n admit at least one $t-$core partition? The results in [3] show that $c_5(n)$ and $c_7(n)$ are positive for all $n \geq 0$. For $t \geq 5$ prime, Olsson has asked if $c_t(n) > 0$ for all n. One easily verifies that $c_2(n)$ and $c_3(n)$ are zero infinitely often. Here are the first few terms of the relevant generating functions.

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} c_2(n)q^n = 1 + q + q^3 + q^6 + q^{10} + q^{15} + q^{21} + \dots$$

$$\prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} = \sum_{n\geq 0} c_3(n)q^n = 1+q+2q^2+2q^4+q^5+2q^6+q^8+2q^9+2q^{10}+2q^{12}+\dots$$

In fact, it is a classical fact that

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} q^{t_n}.$$

Here $t_n = \frac{n(n+1)}{2}$ are the usual triangular numbers.

It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$ [4,6]. We will prove that if $t \geq 4$ and even, then $c_t(n)$ is always positive. Moreover, we will see that for any fixed $t \geq 4$, there are only finitely many n such that $c_t(n) = 0$.

The reader should that $c_t(n) \leq c_{tk}(n)$ for all n. If a partition has no hook numbers divisible by t, then it certainly has no hook numbers divisible by any multiple of t. Hence the conjecture reduces to a study of $c_p(n)$ when p is a prime. For t that are divisble by 2 and 3 we will will use the theory of quadratic forms, modular forms and Gauss' Eureka theorem to prove the positvity of $c_t(n)$. When $p \geq 5$ is prime, we will apply Deligne's theorem on the estimates of Fourier coefficients of cusp forms.

2. An application of Deligne's theorem

In this section we are interested in the modular forms

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \sum_{n\geq \delta_p} a(n)q^n$$

where $p \geq 5$ is prime and $\delta_p = \frac{p^2-1}{24}$. This form is a modular form on $\Gamma_0(p)$ of type $(\frac{p-1}{2}, \epsilon)$ where $\epsilon(n) = \left(\frac{n}{p}\right)$. Clearly, δ_p is the order of this modular form at $i\infty$. These forms do not vanish at $\tau = 0$.

Consequently, since $\Gamma_0(p)$ has only 2 cusps, the cusps at $\tau = i\infty$ and $\tau = 0$, we get the following decomposition

$$\frac{\eta^p(p\tau)}{\eta(\tau)} = \alpha_E E(\tau) + f_{cusp}$$

with $\alpha_E \neq 0$. $E(\tau)$ is the classical Eisenstein series centered at $\tau = 0$ defined by

$$E(\tau) = \sum_{n\geq 1}\sum_{d|n} \epsilon(\frac{n}{d}) d^{\frac{p-3}{2}} q^n.$$

There are no cusp forms on $\Gamma_0(5)$ with character $\epsilon(n) = \left(\frac{n}{5}\right)$. Consequently, it is easy to see that the generating function $\frac{\eta^5(5\tau)}{\eta(\tau)}$ is exactly the Eisenstein series.

In fact, we use these facts to show that if $p \geq 7$, then there must be a cusp form in the decomposition of $\frac{\eta^p(p\tau)}{\eta(\tau)}$ into Eisenstein series and cusp forms. This is true since

$$a(n) = 0 \quad \text{when } 1 \leq n \leq \delta_p - 1.$$

If $p \geq 7$, then $\delta_p \geq 1$. Since the Fourier coefficient of q in $E(\tau)$ is 1, we need a cusp form to cancel it.

Now we return to the conjecture.

Theorem 1. *(Deligne)*
Let $f(\tau) = \sum_{n\geq 1} a(n)q^n$ be a cusp form of weight k for Γ, some congruence subgroup of $SL_2(Z)$, then

$$a(n) = O_\epsilon(n^{\frac{k-1}{2}+\epsilon}) \quad \textit{for all } \epsilon > 0.$$

Using these estimates, it is clear that the Fourier coefficients of the Eisenstein series $E(\tau)$ dominate the Fourier coefficients of a cusp form of the same weight beyond a certain point. Hence, since the Fourier coefficients $a(n)$ of $\frac{\eta^p(p\tau)}{\eta(\tau)}$ are non-negative, it is clear that $\alpha_E > 0$. Consequently, at most finitely many Fourier coefficients $a(n)$ are zero.

This proves the following theorem which is also contained in [6].

Theorem 2. *If t is a positive integer with at least one prime factor $p \geq 5$, then*

$$c_t(n) = 0 \quad \textit{for at most finitely many} \quad n \geq 0.$$

One can verify the positivity of $c_p(n)$ by explicitly calculating the decomposition of $\frac{\eta^p(p\tau)}{\eta(\tau)}$ into Eisenstein series and normalized eigenforms. One can explicitly calculate the point in the Fourier expansion beyond which the Eisenstein series dominates the cusp forms. Hence, for all primes p, the above theorem is easily strengthened by a finite calculation. The author has verified that $c_{11}(n)$ is always positive.

3. Remaining Cases

Deligne's theorem provides a partial solution to the positivity conjecture for all $t \geq 5$ that have prime divisors other than 2 and 3. Here we settle the remaining cases.

In the next theorem we show that $c_t(n)$ is positive when t is a multiple of 4. To do this, it suffices to show that $c_4(n)$ is always positive.

Theorem 3. *If t is a multiple of 4, then $c_t(n) > 0$ for all $n \geq 0$.*

Proof. Consider the $\eta-$ quotient

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = q^5 \sum_{n=0}^{\infty} c_4(n) q^{8n} = \sum_{n=0}^{\infty} c_4(n) q^{8n+5}.$$

The first few terms in its Fourier expansion are

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = q^5 + q^{13} + 2q^{21} + 3q^{29} + q^{37} + \dots.$$

This $\eta-$quotient is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(32)$. The Serre-Stark Basis theorem for modular forms of weight $\frac{1}{2}$ states that all modular forms with weight $\frac{1}{2}$ on $\Gamma_0(4N)$ are linear combinations of theta series. In this case, we find that this $\eta-$quotient is a product of theta series on $\Gamma_0(32)$. The relevant theta series are

$$\Theta_1(\tau) = \frac{1}{2} \sum_{n \equiv 1 \mod 2} q^{n^2} = q + q^9 + q^{25} + q^{49} + \dots$$

and

$$\Theta_2(\tau) = \frac{1}{2} \sum_{n \equiv 1 \mod 2} q^{2n^2} = q^2 + q^{18} + q^{50} + q^{98} + \dots.$$

We obtain the following identity:

$$\frac{\eta^4(32\tau)}{\eta(8\tau)} = \Theta_1(\tau)\Theta_2^2(\tau).$$

Let Q be a ternary quadratic form and let $r(Q, n)$ denote the number of representations of n by the quadratic form Q by non-negative integers. The above identity implies that

$$c_4(n) = r(x^2 + 2y^2 + 2z^2, 8n + 5).$$

This follows from the fact that if $x^2 + 2y^2 + 2z^2$ represents an integer of the form $8n+5$, then x, y and z are all odd. Consequently, we must show that $r(x^2 + 2y^2 + 2z^2, 8n + 5) > 0$ for all $n \geq 0$.

The methods in [5] complete the proof. The genus of the ternary form $x^2 + 2y^2 + 2z^2$ consists of one equivalence class. By Corollary 44a [5], every integer n is represented by some form in the genus of a ternary quadratic form Q if

$$Q \equiv n \mod p^{r+1}$$

is solvable for every prime $p \mid 2d$ where d is the discriminant of Q. If p is odd, then p^r is the highest power of p dividing n. If $p = 2$, then p^r is the highest power of 2 dividing $4n$.

The discriminant of $Q = x^2 + 2y^2 + 2z^2$ is 4. Consquently, every integer of the form $8n + 5$ is represented by Q since the relevant congruence

$$Q = x^2 + 2y^2 + 2z^2 \equiv 8n + 5 \equiv 5 \mod 8$$

is solvable and since the genus of Q is 1 (i.e. Equivalent ternary quadratic forms represent the same integers.). This completes the proof.

□

Now we show that if $t \geq 4$ is even, then $c_t(n)$ is always positive. As a consequence of the last theorem, it suffices to assume that $t = 4s+2$ where $s \geq 1$. This theorem is a corollary to Gauss' famous *Eureka* theorem. This theorem asserts that every non-negative integer can be represented as a sum of 3 triangular numbers. Recall that the n^{th} triangular number is defined by

$$t_n = \frac{n(n+1)}{2}.$$

Here are the first few triangular numbers:

$$t_0 = 0, \quad t_1 = 1, \quad t_2 = 3, \quad t_3 = 6 \quad \text{and } t_4 = 10.$$

Let $\Delta_k(n)$ be the number of representations of n as a sum of k triangular numbers. Define the formal power series generating function for the triangular number t_n by

$$\Psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n=0}^{\infty} q^{t_n}.$$

Consequently, we obtain the following identity of formal power series:

$$\Psi^k(q) = \sum_{n=0}^{\infty} \Delta_k(n) q^n.$$

Now we state the famous *Eureka* theorem.

Theorem 4. *(Gauss)*
If $\Delta_k(n)$ denotes the number of representations of n as a sum of k triangular numbers and $k \geq 3$, then $\Delta_k(n) > 0$ for all $n \geq 0$.

Theorem 5. *If $s \geq 1$ and $t = 4s+2$, then $c_t(n) > 0$ for all $n \geq 0$.*

Proof. We simply manipulate the generating function for $c_t(n)$ by splitting it into a product of two formal power series.

$$\sum_{n=0}^{\infty} c_{4s+2}(n) q^n = \prod_{n=1}^{\infty} \frac{[1-q^{n(4s+2)}]^{4s+2}}{(1-q^n)}$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1-q^{n(4s+2)})^2}{(1-q^{n(2s+1)})} \right]^{2s+1} \prod_{n=1}^{\infty} \frac{(1-q^{n(2s+1)})^{2s+1}}{(1-q^n)}$$

$$= \Psi^{2s+1}((2s+1)q) \sum_{n=0}^{\infty} c_{2s+1}(n) q^n$$

$$= \sum_{n=0}^{\infty} \Delta_{2s+1}(n) q^{n(2s+1)} \sum_{n=0}^{\infty} c_{2s+1}(n) q^n.$$

Since $s \geq 1$, we know that $\Delta_{2s+1}(n) > 0$ for all $n \geq 0$. Furthermore, if $0 \leq n \leq 2s$, then $0 < p(n) = c_{2s+1}(n)$ where $p(n)$ is the usual partition function (i.e. There aren't enough nodes to make a $2s+1$ hook when $0 \leq n \leq 2s$.). It is now easy to see that the product power series has positive coefficients. This completes the proof that $c_{4s+2}(n) > 0$ for all $n \geq 0$.

□

The only values of $t \geq 4$ that we haven't considered are those t that are powers of 3. We complete our discussion of the positivity of $c_t(n)$ by demonstrating that $c_9(n)$ is always positive.

Theorem 6. *If $c_9(n)$ is the number of $9-$core partitions of n, then $c_9(n) > 0$ for all $n \geq 0$.*

Proof. The generating function for $c_9(n)$ is

$$\prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^n)} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3}$$

$$= \sum_{n=0}^{\infty} c_3(n)q^n \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3}.$$

It is well known that $\frac{\eta^9(3\tau)}{\eta^3(\tau)}$ is the normalized weight 3 Eisenstein series on $\Gamma_0(3)$ with character $\epsilon(n) = \left(\frac{n}{3}\right)$ that vanishes at $\tau = i\infty$. Its Fourier expansion is

$$\frac{\eta^9(3\tau)}{\eta^3(\tau)} = q + 3q^2 + 9q^3 + \ldots$$

$$= \sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n)q^n.$$

Here $\sigma_{2,\epsilon}(n)$ is the generalized divisor function defined by

$$\sigma_{2,\epsilon}(n) = \sum_{d|n} \epsilon(\frac{n}{d})d^2.$$

Consequently, we obtain the following identity

$$\frac{\eta^9(9\tau)}{\eta^3(3\tau)} = q^3 \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n)q^{3n}.$$

This identity implies that

$$\prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n)q^{3(n-1)} = \sum_{n=0}^{\infty} \sigma_{2,\epsilon}(n+1)q^{3n}.$$

The coefficients of the last power series are positive because the generalized divisor function $\sigma_{2,\epsilon}(n)$ is always positive.

Combining these facts we obtain

$$\sum_{n=0}^{\infty} c_9(n)q^n = \{1 + q + 2q^2 + \dots\} \sum_{n=0}^{\infty} \sigma_{2,\epsilon}(n+1)q^{3n}.$$

Since the first 3 coefficients of the power series in braces are positive and the generalized divisor function $\sigma_{2,\epsilon}(n)$ is always positive, we find that $c_9(n)$ is always positive. This completes the proof.

□

In summary we have proven the following theorem.

Theorem 7. *Fix an integer $t \geq 4$. If $c_t(n)$ is the number of $t-$core partitions of n, then there are at most finitely many n such that $c_t(n) = 0$. Furthermore, if $t \geq 4$ is even or divisible by 5,7,9 or 11, then $c_t(n)$ is always positive.*

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