A NOTE ON THE NUMBER OF $t-$CORE PARTITIONS

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Abstract. A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. A Ferrers graph represents a partition in the natural way. Fix a positive integer t. A partition of n is called a $t-$*core* partition of n if none of its hook numbers are multiples of t. Let $c_t(n)$ denote the number of $t-$core partitions of n. It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$. In [7], the author proved the conjecture for $t \geq 4$ even and for those t divisible by at least one of $5, 7, 9$, or 11. Moreover if $t \geq 5$ is odd, then it was shown that $c_t(n) > 0$ for n sufficiently large. In this note we show that if $k \geq 2$, then $c_{3k}(n) > 0$ for all n using elementary arguments.

A partition of a positive integer n is a nonincreasing sequence of positive integers with sum n. Here we define a special class of partitions.

Definition 1. Let $t \geq 1$ be a positive integer. Any partition of n whose Ferrers graph have no hook numbers divisible by t is known as a $t-$core partition of n.

The hooks are important in the representation theory of finite symmetric groups and the theory of cranks associated with Ramanujan's congruences for the ordinary partition function [3,4,5].

If $t \geq 1$ and $n \geq 0$, then we define $c_t(n)$ to be the number of partitions of n that are $t-$core partitions. The arithmetic of $c_t(n)$ is studied in [3,4]. The power series generating function for $c_t(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}. \tag{1}$$

One easily verifies that $c_2(n)$ and $c_3(n)$ are zero infinitely often. Here are the first few terms of the relevant generating functions.

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} c_2(n)q^n = 1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\dots$$

$$\prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} = \sum_{n\geq 0} c_3(n)q^n = 1+q+2q^2+2q^4+q^5+2q^6+q^8+2q^9+2q^{10}+2q^{12}+\dots$$

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In fact, it is a classical fact that

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} q^{t_n}.$$

Here $t_n = \frac{n(n+1)}{2}$ are the usual triangular numbers.

Exact formulae for $c_5(n)$ and $c_7(n)$ appear in [3]. Given a positive integer t, is $c_t(n) > 0$ for all $n \geq 0$? In other words, does every positive integer n admit at least one $t-$core partition? The results in [3] show that $c_5(n)$ and $c_7(n)$ are positive for all $n \geq 0$. For $t \geq 5$ prime, Garvan and Olsson have asked if $c_t(n) > 0$ for all n. It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$.

The reader should note that $c_t(n) \leq c_{tk}(n)$ for all n. If a partition has no hook numbers divisible by t, then it certainly has no hook numbers divisible by any multiple of t. Hence the conjecture essentially is reduced to a study of $c_p(n)$ where p is prime. The only obstructions to this method is an analysis of $c_t(n)$ where t is a multiple of 2 or 3; when $t = 2$ or 3 the conjecture is false.

In [7], the author proved the following partial solution to this conjecture using Deligne's estimates on the Fourier coefficients of modular forms, Gauss' Eureka theorem, and quadratic form theory.

Theorem 1. *If $t \geq 4$, then $c_t(n) > 0$ for n sufficiently large. Furthermore, if $t \geq 4$ is even, or divisible by $5, 7, 9,$ or 11, then $c_t(n) > 0$ for all $n \geq 0$.*

The proof of the conjecture when $t \equiv 2 \mod 4$ is an application of Gauss' Eureka Theorem. We now show that similar methods show that $c_{3k}(n) > 0$ for all $n \geq 0$ if $k \geq 2$. First we recall the proof when $t = 9$.

Theorem 2. *If $c_9(n)$ is the number of $9-$core partitions of n, then $c_9(n) > 0$ for all $n \geq 0$.*

Proof. The generating function for $c_9(n)$ is

$$\prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^n)} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3}$$

$$= \sum_{n=0}^{\infty} c_3(n) q^n \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3}. \tag{2}$$

The last infinite product corresponds to a weight 3 Eisenstein series on $\Gamma_0(3)$ with Dirichlet character $\epsilon(n) = \left(\frac{n}{3}\right)$ [6 ,Theorem 6]. This means that its power series expansion is given by the divisor function $\sigma_{2,\epsilon}(n)$ in the following way:

$$\prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n) q^{3(n-1)}. \tag{3}$$

Here the divisor function $\sigma_{2,\epsilon}(n)$ is defined by

$$\sigma_{2,\epsilon}(n) = \sum_{0<d|n} \epsilon\left(\frac{n}{d}\right) d^2. \tag{4}$$

It is an easy exercise to verify that all of the coefficients in (3) are positive since $\sigma_{2,\epsilon}(n) > 0$ for all $n \in \mathbb{Z}^+$.

Combining these facts we obtain from (2) and (3)

$$\sum_{n=0}^{\infty} c_9(n)q^n = \{1 + q + 2q^2 + \dots\} \sum_{n=0}^{\infty} \sigma_{2,\epsilon}(n)q^{3(n-1)}.$$

Since the first 3 coefficients of the power series in braces are positive and the generalized divisor function $\sigma_{2,\epsilon}(n)$ is always positive, we find that $c_9(n)$ is always positive. This completes the proof.

□

It should be noted that Fine [2, 3.2.351, p. 79] has an elementary proof of this fact.

Now we prove the main theorem of this note using Theorem 2.

Theorem 3. *If $k \geq 2$, then $c_{3k}(n) > 0$ for all $n \geq 0$.*

Proof. If $k \equiv 0 \mod 3$, then $9 \mid 3k$. By Theorem 2 we find that $0 < c_9(n) \leq c_{3k}(n)$ for all $n \geq 0$. Therefore we may assume that $k \not\equiv 0 \mod 3$.

We may assume that $k = 3t + i$ with $i = 1$ or 2. The generating function for $c_{3k}(n) = c_{9t+3i}(n)$ can be factored in the following way:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{9t+3i}(n)q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{9t+3i}}{(1-q^n)} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{9t}}{(1-q^{(3t+i)n})^{3t}} \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{3i}(1-q^{(3t+i)n})^{3t}}{(1-q^n)} \\
&= \left[\sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n)q^{(3t+i)(n-1)}\right]^t \prod_{n=1}^{\infty} \frac{(1-q^{(3t+i)n})^{3t+i}}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{3i}}{(1-q^{(3t+i)n})^{i}} \\
&= \left[\sum_{n=1}^{\infty} \sigma_{2,\epsilon}(n)q^{(3t+i)(n-1)}\right]^t \left[\sum_{n=0}^{\infty} c_{3t+i}(n)q^n\right] \left[\sum_{n=0}^{\infty} c_3(n)q^{(3t+i)n}\right]^i. \qquad (5)
\end{aligned}$$

Since $\sigma_{2,\epsilon}(n) > 0$ for all $n \geq 1$, we see that the the first factor of (5), the tth power of the divisor function power series, has positive coefficients for exponents that are multiples of $3t + i$. The coefficients $c_{3t+i}(n)$ of the middle factor in (5) are positive for all $0 \leq n \leq 3t + i - 1$; one needs at least $3t + i$ nodes before a partition can have a $3t + i$ hook. Therefore the product of the first two factors in (5) has nothing but positive coefficients. Since $c_3(0) = 1$, we find that the coefficients of the entire product, namely $c_{9t+3i}(n)$ are all positive.

□

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References

1. G. Andrews, *Eureka! num* $= \Delta + \Delta + \Delta$, J. Number theory **23** (1986), 285-293.

2. N.J. Fine, *Basic hypergeometric series and applications*, American Mathematical Society, 1988.

3. F. Garvan D. Kim and D. Stanton, *Cranks and $t-$cores*, Invent. Math. **101** (1990).

4. F. Garvan, *Some congruence properties for partitions that are $t-$cores*, Proc. London Math. Soc. **66** (1993).

5. A. Klyachko, *Modular forms and representations of symmetric groups, integral lattices and finite linear groups*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov **116** (1982).

6. K. Ono, *Congruences on the Fourier coefficients of modular forms on* $\Gamma_0(N)$, Ph.D. Thesis, The University of California, Los Angeles, 1993.

7. ______, *On the positivity of the number of $t-$core partitions*, Acta Arithmetica **to appear**.

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