A NOTE ON THE SHIMURA CORRESPONDENCE AND THE RAMANUJAN $\tau(n)$ FUNCTION

KEN ONO

ABSTRACT. The Shimura correspondence is a family of maps which sends modular forms of half-integral weight to forms of integral weight by a Mellin transform [9]. Prior to Shimura's work, Selberg discovered a special case of this correspondence when the half-integral weight form is a cusp eigenform on $SL_2(Z)$ times the classical theta function. In a recent paper [1], Cipra generalizes Selberg's method and explicitly demonstrates the image of certain Shimura maps for half-integral weight forms that are newforms of type (k, χ) on $\Gamma_0(N)$ times a theta series. In this note, we apply this lift twice to a newform with complex multiplication by $K = Q(i)$, and we obtain a formula for the Ramanujan function $\tau(n)$ in terms of the arithmetic of ideals in the ring O_K via a Hecke character. These observations are connected to Lehmer's conjecture on the nonvanishing of $\tau(n)$, the representations of integers as sums of 5 squares, and affine root systems of simple Lie algebras.

1. CIPRA'S THEOREM

In this section we state Cipra's theorem which explicitly describes the image of a particular Shimura map applied to a newform times a theta series. We borrow Cipra's notation for its clarity.

Let χ be a Dirichlet character $\bmod 4N$, and let t be a positive square-free positive integer. Let $F(\tau) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(4N, \chi)$ where $k \in Z^+$. Define $A_t(n)$ by the following identity:

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = L(s-k+1, \chi_t^{(k)}) \sum_{m=1}^{\infty} \frac{b(tm^2)}{m^s},$$

where $\chi_t^{(k)}(m) = \chi(m)\left(\frac{-1}{m}\right)^k\left(\frac{t}{m}\right)$ is a Dirichlet character $\bmod 4Nt$.

The image of F under the Shimura map S_t is defined by

$$S_t(F) = \sum_{n=1}^{\infty} A_t(n)q^n.$$

The essence of the correspondence is that half-integral weight cusp forms correspond to integral weight cusp forms when $k > 1$. Specifically, if $k > 1$, then $S_t(F) \in S_{2k}(2N, \chi^2)$. If $k = 1$, then $S_t(F) \in M_{2k}(2N, \chi^2)$.

Now we state a case of Cipra's generalization of Selberg's method that explicitly lifts half-integral weight cusp forms that are products of a newform by a theta series.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Theorem 1.0.1. *(Cipra) Let $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \chi)$ be a newform of type (k, χ). Define $\Theta(\tau)$, a modular form with weight $\frac{1}{2}$ on $\Gamma_0(4)$, by*

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + \ldots .$$

Let $F(\tau) \in S_{k+\frac{1}{2}}(4N, \chi_1^{(k)})$ be defined by

$$F(\tau) = \sum_{n=4}^{\infty} b(n)q^n = f(4\tau)\Theta(\tau).$$

Then the image of $F(\tau)$ by the Shimura map S_1 is

$$S_1(F(\tau)) = f^2(\tau) - 2^{k-1}\chi(2)f^2(2\tau).$$

Furthermore, $S_1(F(\tau)) \in S_{2k}(2N, \chi^2)$.

This theorem says that the image of $f(4\tau)\Theta(\tau)$ is essentially $f^2(\tau)$. Note that this theorem has been generalized to the case of Hilbert modular forms by Shemanske and Walling [8].

2. New formulas

Recall that the Ramanujan function $\tau(n)$ is defined by the Fourier expansion of $\Delta(\tau)$, the unique normalized cusp form of weight 12 on $SL_2(Z)$. In particular, we have

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.$$

In this section, we apply the theorem of Cipra to establish a formula for $\tau(n)$. Due to the Hecke multiplicative nature of eigenforms, we only need to establish formulas for $\tau(p)$, where p is a prime.

To accomplish our goal, we will need the Dedekind $\eta-$function, a modular form of weight $\frac{1}{2}$. It is defined by the infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n).$$

Note that the form $\Delta(\tau)$ is just the 24^{th} power of $\eta(\tau)$. We are interested in the newforms $\eta^6(4\tau)$ and $\eta^{12}(2\tau)$.

First we will apply the Shimura map S_1 to $\eta^6(4\tau) \in S_3(16, \chi)$ where $\chi(d) = \left(\frac{-1}{d}\right)$. Here are the first few terms of its Fourier expansion:

$$\eta^6(4\tau) = q\prod_{n=1}^{\infty}(1 - q^{4n})^6 = q - 6q^5 + 9q^9 + 10q^{13} - \ldots .$$

This newform is a form with complex multiplication in the sense of [6].

Here we recall the construction of such newforms. Let $K = Q(\sqrt{-d})$ be a quadratic imaginary field, and let O_K be its ring of algebraic integers. A Hecke character ϕ of weight $k \geq 2$ with modulus Λ is defined in the following way. Let Λ be a nontrivial ideal in O_K and let $I(\Lambda)$ denote the group of fractional ideals prime to Λ. A Hecke character ϕ with modulus Λ is homomorphism

$$\phi : I(\Lambda) \rightarrow C^*$$

satisfying

$$\phi(\alpha O_K) = \alpha^{k-1} \quad \text{when} \quad \alpha \equiv 1 \mod \Lambda.$$

The series $\Psi(\tau)$ induced by the Hecke character ϕ defined by

$$\Psi(\tau) = \sum_{a} \phi(a) q^{Na} = \sum_{n=1}^{\infty} a(n) q^n,$$

where the sum is over the integral ideals a that are prime to Λ and Na is the norm of the ideal a, is a modular form of weight k on $\Gamma_1(| d | N\Lambda)$. Such a form is known as a modular form with complex multiplication by K. An important feature to notice here is that if $p \in Z$ is prime that is inert in K, then the Fourier coefficient $a(p) = 0$ because there are no ideals with norm p. There are other interesting properties satisfied by these forms. The reader may consult [2],[5],[7].

The form $\eta^6(4\tau) = \sum_{n=1}^{\infty} h(n) q^n$ is a form with complex multiplication by $K = Q(i)$. In this case $\Lambda = (2)$ and $k = 3$. Notice that if $p \equiv 3 \mod 4$ is a prime, then p is inert in K. Consequently, if $p \equiv 3 \mod 4$ is a prime, then $h(p) = 0$. Furthermore, suppose that $p \equiv 1 \mod 4$ is prime, then the principal ideal (p) factors as

$$(p) = (x + iy)(x - iy) \quad \text{with} \quad x, y \in Z.$$

Consequently, we find $h(p)$ to be

$$h(p) = \phi((x + iy)) + \phi((x - iy)) = 2x^2 - 2y^2.$$

By Hecke multiplicativity we are now able to calculate all of the Fourier coefficients of $\eta^6(4\tau)$ by studying the splitting of primes in $Q(i)$.

Now we apply Cipra's theorem to our form $\eta^6(4\tau)$. We obtain the following identity:

$$S_1(\eta^6(16\tau)\Theta(\tau)) = \eta^{12}(4\tau).$$

The Shimura lift in this case produces essentially the square root of $\Delta(\tau)$. The form $\eta^{12}(2\tau) = \sum_{n=1}^{\infty} s(n) q^n$ is a newform of weight 6 on $\Gamma_0(4)$. Consequently we have

$$S_1(\eta^6(16\tau)\Theta(\tau)) = \sum_{n=1}^{\infty} s(n) q^{2n}.$$

Let $\eta^6(16\tau)\Theta(\tau) = \sum_{n=4}^{\infty} b(n) q^n$. By definition, S_1 is the Mellin transform of $L(s-2, \chi) \sum_{m=1}^{\infty} \frac{b(n^2)}{n^s}$. So we find that if p is a prime, then $s(p)$ corresponds to the term with $(2p)^{-s}$ in the $L-$series. Consequently, if p is an odd prime, then

$$s(p) = b(4p^2) + \chi(p) p^2 b(4) = b(4p^2) + \chi(p) p^2. \tag{2.1}$$

Note that $s(2) = 0$. We now only need to calculate $b(4p^2)$. It is easy to verify that $b(4p^2) = h(p^2) + 2\sum_{k=1}^{[\frac{p}{2}]} h(p^2 - 4k^2)$, where $[x]$ is the greatest integer function. Here we used the fact that $h(2n) = 0$. Consequently, we obtain the following formula for the Fourier coefficients of $\eta^{12}(2\tau)$.

Lemma 2.0.2. *With the given notation we find that if p is an odd prime, then*

$$s(p) = h(p^2) + \chi(p) p^2 + 2 \sum_{k=1}^{[\frac{p}{2}]} h(p^2 - 4k^2).$$

Furthermore $s(2) = 0$.

Since $h(n)$ is determined by the Hecke character ϕ, we find that the Fourier expansion of $\eta^{12}(2\tau)$ is also a function of ϕ. We will explain this in greater detail momentarily.

Applying Cipra's theorem to the newform $\eta^{12}(2\tau)$ yields similar results. In fact, we obtain the following identity:

$$S_1(\eta^{12}(8\tau)\Theta(\tau)) = \Delta(2\tau). \tag{2.2}$$

By arguments similar to those used earlier, we obtain the following formulas.

Lemma 2.0.3. *Let $\tau(n)$ be the usual Ramanujan function. If p is an odd prime, then*

$$\tau(p) = s(p^2) + p^5 + 2\sum_{k=1}^{[\frac{p}{2}]} s(p^2 - 4k^2).$$

Furthermore, $\tau(2) = -24$.

Since the Fourier coefficients of eigenforms of Hecke operators are multiplicative (i.e. $a(m)a(n) = a(mn)$ when $(m,n) = 1$), we only need to know how to calculate the Fourier coefficients associated with prime power exponents. If p is a prime, then the action of a Hecke operator T_p, on an eigenform $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ of type (k,χ), yields the following recurrance:

$$a(p^{r+1}) = a(p)a(p^r) - \chi(p)p^{k-1}a(p^{r-1}). \tag{2.3}$$

This recursion allows us to view $a(p^r)$ as a function of $a(p)$. In particular, define $C_{n,k,\chi}(x,p)$ to be the generalized n^{th} Chebyshev polynomial dependent on k and χ. These polynomials are defined in the following way:

$$C_{1,k,\chi}(x,p) = x, \quad C_{2,k,\chi}(x,p) = x^2 - \chi(p)p^{k-1},$$

$$\text{and} \quad C_{r+1,k,\chi}(x,p) = xC_{r,k,\chi}(x,p) - \chi(p)p^{k-1}C_{r-1,k,\chi}(x,p).$$

These polynomials allow us to calculate the Fourier coefficients of a normalized eigenform. In particular, suppose that $n = \prod_{i=1}^{j} p_i^{r_i}$ is the prime factorization of n. Then it is easy to see that

$$a(n) = \prod_{i=1}^{j} C_{r_i,k,\chi}(a(p_i), p_i).$$

Hence, the Fourier expansion of f are uniquely determined by the Fourier coefficients associated with prime exponents. Using these polynomials we obtain the following formulas for $s(p)$ and $\tau(p)$ when p is an odd prime.

Lemma 2.0.4. *Let p be an odd prime. For all $1 \le k \le [\frac{p}{2}]$, we denote the factorizations of $p^2 - 4k^2$ by*

$$p^2 - 4k^2 = \prod_{i=1}^{m_k} p_{i,k}^{s_{i,k}}.$$

Using the notation defined above, we obtain

$$s(p) = C_{2,3,(\frac{-1}{n})}(h(p),p) + \left(\frac{-1}{p}\right)p^2 + 2\sum_{k=1}^{[\frac{p}{2}]}\prod_{i=1}^{m_k} C_{s_{i,k},3,(\frac{-1}{n})}(h(p_{i,k}), p_{i,k}) \tag{2.4}$$

and

$$\tau(p) = C_{2,6,Id}(s(p), p) + p^5 + 2\sum_{k=1}^{[\frac{p}{2}]}\prod_{i=1}^{m_k} C_{s_{i,k},6,Id}(s(p_{i,k}), p_{i,k}). \quad (2.5)$$

In the above proposition we are able to express $s(n)$ and $\tau(n)$ in terms of $h(p)$ where $h(p)$ is defined by the Hecke character ϕ.

At this point we can consider Lehmer's conjecture that $\tau(n)$ is never 0. Of course, there is also the related conjecture that $s^*(n)$ is never 0, where $s^*(n)$ is defined by

$$\prod_{n=1}^{\infty}(1-q^n)^{12} = \sum_{n=0}^{\infty} s^*(n)q^n.$$

It is easy to verify that $s^*(n) = s(2n+1)$. Therefore, the related conjecture says that $s(2n+1)$ never vanishes. Recall that $h(p) = 0$ for all primes $p \equiv 3 \mod 4$. Consequently, one may wonder if there exists a prime such $p \equiv 3 \mod 4$ such that $s(p)$ or $\tau(p)$ vanish by virtue of the inertness of primes in $Q(i)$. In other words, is there an obvious counterexample for Lehmer's conjecture or the related conjecture?

It is easy to verify that if $p \equiv 3 \mod 4$ is prime, then $h(p^s) = 0$ when s is odd. Furthermore, it is also easy to verify that $h(p^2) = p^2$. Consequently, the first 2 terms in the formula for $s(p)$ cancel. If $p \equiv 3 \mod 4$, then

$$s(p) = 2\sum_{k=1}^{[\frac{p}{2}]}\prod_{i=1}^{m_k} C_{s_{i,k},3,(\frac{-1}{n})}(h(p_{i,k}), p_{i,k}). \quad (2.6)$$

We are then led to search for a prime $p \equiv 3 \mod 4$ such that for all $1 \le k \le [\frac{p}{2}]$, at least one prime factor $p_{i,k}$ of $p^2 - 4k^2$ satisfies $p_{i,k} \equiv 3 \mod 4$ with $s_{i,k}$ is odd. If such a prime p exists , then $s(p) = 0$ showing that the conjecture is false. Furthermore, this argument would provide clues regarding Lehmer's conjecture.

However no such prime p exists. It is well known that an integer n is representable as a sum of two squares if and only if all prime divisors q of n occur in n to an even power. So we want $p^2 - 4k^2$ to be unrepresentable as a sum of 2 squares for all k. In other words, there are no solutions to

$$p^2 = x^2 + y^2 + 4y^2. \quad (2.7)$$

By Lagrange, there exist integers a, b, c, and d such that

$$p = a^2 + b^2 + c^2 + d^2.$$

By Lebesgue's identity, we obtain

$$p^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2. \quad (2.8)$$

Hence, there is no such prime p. Consequently, there is no trivial way coming from the splitting of primes in $Q(i)$ that forces $s(p) = 0$. $s(p) = 0$ is still possible but it just does not come for free. For similar reasons, Lehmer's conjecture remains open.

3. Additional remarks

Here we take the opportunity to examine another formula for $\tau(n)$. In [4], MacDonald proves combinatorial identities for powers of the Dedekind $\eta-$function induced by specializing formulas arising from the study of affine root systems on Lie algebras. MacDonald proves the following formula for $\tau(n)$.

Theorem 3.0.5. *Let V be a subset of Z^5 consisting of vectors $v=(v_0,v_1,v_2,v_3,v_4)$ satisfying*

$$\sum_{i=0}^{4} v_i = 0 \quad \text{and} \quad v_i \equiv i \mod 5.$$

Let $N(v)=\sum_{i=0}^{4} v_i^2$. Given these hypotheses,

$$\tau(n) = \frac{1}{288} \sum_{v\in V,\ Nv=10n} \prod_{i<j} (v_i - v_j).$$

So, we are now interested in the representations of $10n$ as the sum of 5 squares with some restrictions. It is interesting to see that this combinatorial formula for $\tau(n)$ is equivalent to the field theoretic one using the Hecke character ϕ. Just as a side note, Stieljtes was the first the derive a formula for $r_5(n)$, the number of unrestricted representations of n as a sum of 5 squares [3], [11]. If p is an odd prime, he proved

$$r_5(p) = 10\left\{\sigma(p^2) + 2\sum_{k=1}^{[\frac{p}{2}]} \sigma(p^2-4k^2)\right\}.$$

Here $\sigma(n)$ is defined by $\sigma(n)=\sum_{d|n, d \text{ odd}} d$. I am noting this formula because of its similarity to the Shimura map formulas proven in Lemmas 1 and 2.

Incidentally, many of the identities in [4] correspond to $\eta-$products with complex multiplication. Consequently, the vectors which determine Fourier coefficients correspond nicely to the Hecke characters which define the same coefficients. This is the topic of a future paper.

4. Acknowledgements

The author would like to thank Bruce Abe for many calculations which provided evidence for the content of this paper. The author would also like to thank Basil Gordon for many helpful suggestions.

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Department of Mathematics, The University of Georgia, Athens, Georgia 30602
E-mail address: ono@joe.math.uga.edu