# CONGRUENCES FOR FROBENIUS PARTITIONS

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Abstract. The partition function $p(n)$ has several celebrated congruence properties which reflect the action of the Hecke operators on certain holomorphic modular forms. In this article similar congruences are proved for $c_3(n)$, the number of generalized Frobenius partitions of $n$ with 3 colors. We prove

$$c_3(63n+50) \equiv 0 \mod 7, \tag{1}$$

$$c_3(5n+2) \equiv p\left(\frac{5n+2}{3}\right) \mod 5, \tag{2}$$

except when $n = 3T_m$ and $T_m = \frac{m(m+1)}{2}$ is the $m^{th}$ triangular number, and

$$c_3(15T_m+2) \equiv (-1)^m(m+3) \mod 5. \tag{3}$$

Congruences (2) and (3) are analogous to Euler's pentagonal number theorem. These congruences are proved by constructing holomorphic modular forms which inherit related congruence properties which are verified computationally via Sturm's criterion.

## 1. Introduction

A partition of a non-negative integer $n$ is a non-increasing sequence of integers whose sum is $n$. The number of partitions of $n$ is denoted by $p(n)$. The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{4}$$

A *generalized Frobenius partition* is a two rowed array of non-negative integers of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ b_1 & b_2 & b_3 & \dots & b_k \end{pmatrix}$$

where the entries in each row are in non-increasing order. The integer $n$ that is partitioned is $n = \sum_{i=1}^{k}(a_i + b_i + 1)$. A generalized Frobenius partition in 3 colors

1991 *Mathematics Subject Classification.* Primary 05A17 ; Secondary 11P83.
*Key words and phrases.* modular forms, partition congruences.

is an array of the above form where the integer entries are taken from 3 distinct copies of the non-negative integers distinguished by color, and the rows are ordered first by size and then by color with no two consecutive like entries in any row. We let $c_3(n)$ denote the number of generalized Frobenius partitions of $n$ in 3 colors.

**Example.** *In this example we list all of the generalized Frobenius partitions of* 3 *in the three colors blue, green, and red. We denote the color of an integer by the subscript* $b, g, r$ *and we order the colors by* $b < g < r$. *The only partitions consisting of one column, up to choices of colors, are of the form*

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*In each of these cases we have 3 choices of color for each of the two entries from which we obtain 27 partitions of 3.*

*The only partitions of* 3 *with 2 columns, up to choices of colors, are*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

*In each of these partitions of* 3 *there are* 9 *choices for the coloring of the row* $1 \;\; 0$. *The row* $0 \;\; 0$ *has the* 3 *colorings*

$$0_r 0_b, \quad 0_r 0_g, \quad \text{and} \quad 0_g 0_b$$

*since these rows are ordered by color and no row has two consecutive like entries. Hence there are 54 partitions of* 3 *with 2 columns.*

*The only partition of 3 with 3 columns is*

$$\begin{pmatrix} 0_r & 0_g & 0_b \\ 0_r & 0_g & 0_b \end{pmatrix}$$

*since each row is ordered by color with no consecutive like entries. Since there are no such partitions of 3 with more than 3 columns it follows that* $c_3(3) = 82$.

In [5] Kolitsch proves that the generating function for $\bar{c_3}(n) := c_3(n) - p(n/3)$ is

$$\sum_{n=0}^{\infty} \bar{c_3}(n) q^n = 9q \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^{3n})(1-q^n)^3} = 9q + 27q^2 + 81q^3 + 207q^4 + \ldots \tag{5}$$

( $p(n/3) = 0$ if $n/3$ is not an integer) and uses it to prove Ramanujan-type congruences modulo powers of 3 for $\bar{c_3}(n)$. These congruences are natural analogs of congruences proved by Kolitsch modulo powers of 5 and 7 for other colored generalized Frobenius partitions. In this paper we prove:

**Main Theorem 1.** *In the notation above, the function* $c_3(n)$ *satisfies*

$$c_3(63n + 50) \equiv 0 \mod 7.$$

**Main Theorem 2.** *In the notation above, the function* $c_3(n)$ *satisfies*

$$c_3(5n + 2) \equiv p\left(\frac{5n+2}{3}\right) \mod 5$$

*except when* $n = 3T_m$ *for some* $m$ *where* $T_m := \frac{m(m+1)}{2}$ *is the* $m^{th}$ *triangular number. For such* $n$ *we obtain*

$$c_3(15T_m + 2) \equiv (-1)^m (m+3) \mod 5.$$

## 2. Preliminaries

If $N$ is a positive integer, then we let $\Gamma_0(N)$ and $\Gamma_1(N)$ denote the congruence subgroups of $SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ c \equiv 0 \mod N \right\},$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\quad a \equiv d \equiv 1 \mod N,\ c \equiv 0 \mod N \right\}.$$

The index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$ is

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1})$$

where the product is over primes $p$ dividing $N$. These groups act naturally on the upper half complex plane. In particular, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $\Gamma$ is any of these congruence subgroups, then $Az := \frac{az+b}{cz+d}$. If $k$ is an integer, then any function $f(z)$ on the upper half complex plane satisfying

$$f(Az) = (cz + d)^k f(z) \tag{6}$$

for all $A \in \Gamma$ and all $z \in \mathfrak{H} = \{z | Im(z) > 0\}$ is called a *modular form* of weight $k$ with respect to $\Gamma$. If $f(z)$ is holomorphic on $\mathfrak{H}$ and the cusps (i.e. rationals) of $\Gamma$, then $f(z)$ is known as a *holomorphic modular form.* A holomorphic modular form $f(z)$ which vanishes at all of the cusps of $\Gamma$ is called a *cusp form.*

Of particular interest are certain holomorphic modular forms with respect to the group $\Gamma_1(N)$. Let $\chi$ be a Dirichlet character $\mod N$. Then a holomorphic modular form $f(z)$ on $\Gamma_1(N)$ which satisfies

$$f(Az) = \chi(d)(cz + d)^k f(z) \tag{7}$$

for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ is called a holomorphic modular form with *Nebentypus character* $\chi$. We denote the finite dimensional $\mathbb{C}-$vector space of these holomorphic modular forms of weight $k$ and character $\chi$ with respect to $\Gamma_0(N)$ by $M_k(N, \chi)$. The subspace of cusp forms is denoted $S_k(N, \chi)$.

In the variable $q = e^{2\pi i z}$, a function $f(z) \in M_k(N, \chi)$ admits a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

The Hecke operators are linear transformations acting on Fourier expansions which preserve $M_k(N, \chi)$ and $S_k(N, \chi)$.

If $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(N, \chi)$, then the image under the Hecke operator $T_m$ is $T_m | f(z) = \sum_{n=0}^{\infty} b(n) q^n \in M_k(N, \chi)$ where

$$b(n) = \sum_{d|(m,n)} \chi(d) d^{k-1} a(nm/d^2). \tag{8}$$

Proving the congruences in this paper requires the following proposition on *twists* of Fourier expansions.

**Proposition 1.** [4, p.127] *Let $\chi_1$ and $\chi_2$ be two Dirichlet characters* $\mod N$ *and* $\mod M$ *respectively. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(N, \chi_1)$, then $f_2(z) = \sum_{n=0}^{\infty} \chi_2(n)a(n)q^n \in M_k(NM^2, \chi_1\chi_2^2)$. Moreover, if $f(z)$ is a cusp form, then so is $f_2(z)$.*

Dedekind's Eta-function defined by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

is a holomorphic modular form of weight $\frac{1}{2}$ which does not vanish on $\mathfrak{H}$. A function $f(z)$ is called an Eta-product if it is expressible as a finite product of the form

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z)$$

where $r_\delta \in \mathbb{Z}$. The following theorem due to Gordon, Hughes, and Newman [3,8] describes the modular properties of an Eta-product.

**Theorem 1.** *(**Gordon, Hughes, Newman**) Let $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ be an Eta-product with $r_\delta \in \mathbb{Z}$. If there exists a positive $N$ such that*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \mod 24 \tag{9}$$

*and*

$$\sum_{\delta | N} \frac{N r_\delta}{\delta} \equiv 0 \mod 24, \tag{10}$$

*then $f(z)$ satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

*for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ where $k = \frac{1}{2}\sum_{\delta|N} r_\delta$. Here the character $\chi$ is defined by $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$ and $s = \prod_{\delta|N} \delta^{r_\delta}$.*

In particular, if $k$ is a positive integer and $f(z)$ is holomorphic (resp. vanishes) at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(N, \chi)$ (resp. $S_k(N, \chi)$).

In [7] Ligozat calculated the analytic orders of an Eta-product at the cusps of $\Gamma_0(N)$. Let $c, d$ and $N$ be positive integers with $d \mid N$ and $(c, d) = 1$. Then in the notation above, if the Eta-product $f(z)$ satisfying (9) and (10), then the order of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{1}{24} \sum_{\delta | N} \frac{N(d, \delta)^2 r_\delta}{(d, \frac{N}{d}) d\delta}. \tag{11}$$

When proving congruences, it is often important to determine when the Fourier expansion of a modular form has coefficients that are all multiples of $M$. First we define the $\ell-$adic order of a formal power series.

**Definition.** Let $M$ be a positive integer and $f = \sum_{n\geq N} a(n)q^n$ a formal power series in the variable $q$ with rational integer coefficients. The $M-$adic order of $f$ is defined by

$$Ord_M(f) = \inf\{\ n |\ a(n) \not\equiv 0 \pmod M\}.$$

In [9] Sturm proved the following expicit criterion for determining whether two modular forms are congruent; this reduces the proof of a conjectured congruence to a finite calculation.

**Sturm's Criterion.** *Let $f(z)$ and $g(z)$ be holomorphic modular forms of weight $k$ with respect to some congruence subgroup $\Gamma$ of $SL_2(\mathbb{Z})$. If $f(z)$ and $g(z)$ have rational integer Fourier coefficients and there exists a prime $\ell$ such that*

$$Ord_\ell(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma],$$

*then $Ord_\ell(f(z) - g(z)) = \infty$. (i.e. $f(z) \equiv g(z) \pmod \ell$.)*

The difficulty one encounters when trying to prove congruences for Fourier coefficients of nonholomorphic modular forms is that the theory of Hecke operators and Sturm's Criterion do not apply. Consequently it is desirable to construct holomorphic modular forms which inherit the alleged congruence properties. To accomplish this, it is necessary to construct holomorphic modular forms whose Fourier expansions are congruent to $1 \pmod M$.

Using Eta-products, one can construct numerous examples of holomorphic modular forms whose expansions are congruent to 1 modulo a prime $p$. Here is an easy proposition which provides infinitely many Eta-products with this property.

**Proposition 2.** *If $p \geq 5$ is prime, then the Eta-product*

$$f_k(z) = \frac{\eta^{p^k}(z)}{\eta(p^k z)} \in M_{\frac{p^k-1}{2}}(p^k, \chi_{p,k})$$

*where $\chi_{p,k}(d) = \left(\frac{(-1)^{\frac{p^k-1}{2}} p^{-k}}{d}\right)$. Moreover, $f(z) \equiv 1 \pmod p$.*

*Proof.* By Theorem 1, it turns out that $f_k(z)$ is a modular form on $\Gamma_0(p^k)$ with the given character $\chi_{p,k}$ since $p^{2k} - 1 \equiv 0 \pmod{24}$ when $p \geq 5$ is prime. To show that $f_k(z)$ is a holomorphic modular form it suffices to check that $f_k(z)$ is holomorphic at the cusps of $\Gamma_0(p^k)$. The representatives of the cusps of $\Gamma_0(p^k)$ can be chosen from the set of rationals $\frac{c}{p^a}$ where $0 \leq a \leq k$. By (11) one can verify that the orders at the cusps of $\Gamma_0(p^k)$ is never negative. The congruence follows from the Children's Binomial theorem.

□

We now present an elementary proposition which shows that conjectured congruences for a power series $f$ can be inherited by a power series $fg$ and can be proved by establishing the congruence for the series $fg$.

**Proposition 3.** *Let $f = \sum_{n=0}^{\infty} a(n)q^n$ and let $g = 1 + \sum_{n=1}^{\infty} b(mn)q^{mn}$. Define $c(n)$ by $fg = \sum_{n=0}^{\infty} c(n)q^n$. Let $d$ be a residue class $\bmod m$. If $c(mn + d) \equiv 0 \pmod N$ for all $n$, then $a(mn + d) \equiv 0 \pmod N$ for all $n$.*

*Proof.* It is clear that $c(d) = a(d)$, therefore $a(d) = c(d) \equiv 0 \mod N$ If $n > 1$, then

$$c(mn+d) = a(mn+d) + \sum_{k=1}^{n} b(mk)a(m(n-k)+d).$$

By induction it is clear that the congruence $c(mn+d) \equiv 0 \mod N$ implies the congruence $a(mn+d) \equiv 0 \mod N$.

□

## 3. The congruence for $c_3(n) \mod 7$

In this section we apply these methods to prove (1).

**Theorem 2.** *Define $a(n)$ by the infinite product*

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{3n})(1-q^n)^3}.$$

*Then the function $a(n)$ satisfies the congruence*

$$a(n) \equiv 0 \mod 7 \quad \text{if} \quad n \equiv 22, 40, 49, 58 \mod 63.$$

*Proof.* Define the following modular form

$$f(z) := \frac{\eta^{17}(63z)\eta^3(189z)}{\eta^3(z)\eta(3z)} \left(\frac{\eta^7(z)}{\eta(7z)}\right)^2 = \sum_{n\geq 68} b(n)q^n.$$

By Theorem 1 and (11) it turns out that $f(z) \in S_{14}(189, Id)$ where $Id$ is the trivial Dirichlet character $\mod 189$. By Proposition 2, the coefficients of the first factor possess the congruence properties of the Fourier coefficients of $f(z) \mod 7$.

By Proposition 3 to prove the congruences for $a(n)$ is suffices to prove

$$b(n) \equiv 0 \mod 7 \quad \text{if} \quad n \equiv 0, 27, 45, 54 \mod 63. \tag{12}$$

We note that these residue classes $\mod 63$ are all multiples of 9. Moreover the missing residue classes $n$ which are multiples of 9, $n = 9, 18, 36 \mod 63$ all have the property that $\left(\frac{n}{7}\right) = 1$.

Now we apply Proposition 1 and construct a modular form $f_2(z) \in S_{14}(9261, Id)$ by

$$f_2(z) := \sum_{n\geq 68} \left(\frac{n}{7}\right) b(n)q^n.$$

Define another modular form $F(z) \in S_{14}(9261, Id)$ by

$$F(z) := f(z) - f_2(z) = \sum_{\left(\frac{n}{7}\right)=-1} 2b(n)q^n + \sum_{n\equiv 0 \mod 7} b(n)q^n.$$

If $F(z) = \sum_{n\geq 68} c(n)q^n$, then $c(n) = 0$ for all $n \equiv 9, 18, 36 \mod 63$. Now apply the Hecke operator $T_9$ to $F(z)$ to obtain

$$T_9|F(z) = \sum_{n\geq 1} c(9n)q^n.$$

If $T_9|F(z) \equiv 0 \mod 7$, then (12) is holds. Here are the first few terms of $T_9|F(z)$ :

$$T_9|F(z) = 24094q^{10} + 61558q^{12} - 244090q^{13} + \ldots.$$

One finds that the first $9 \cdot \frac{14}{12} \cdot [SL_2(\mathbb{Z}) : \Gamma_0(9261)] + 1 = 148147$ terms are multiples of 7 which completes the proof by an immediate application of Sturm's Criterion.

□

**Main Theorem 1.** *If $c_3(n)$ denotes the number of Frobenius partitions of $n$ in 3 colors, then*
$$c_3(63n+50) \equiv 0 \mod 7.$$

*Proof.* By (5), we obtain the following factorization for the generating function for $\bar{c_3}(n)$ :
$$\sum_{n=0}^{\infty} \bar{c_3}(n)q^n = 9q \prod_{n=1}^{\infty}(1-q^{9n})^3 \sum_{n=0}^{\infty} a(n)q^n.$$
Recall the following modified formula due to Jacobi:
$$q\prod_{n=1}^{\infty}(1-q^{9n})^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{9n^2+9n}{2}+1}.$$

The only residues of $\frac{9n^2+9n}{2}+1 \mod 63$ are $1, 10, 28, 55 \mod 63$. In the above product, the only coefficients $a(m)$ which contribute to $\bar{c_3}(63n+50)$ are those where $n \equiv 49, 40, 22, 58 \mod 63$. However these residue classes are precisely those classes where $a(n) \equiv 0 \mod 7$ by Theorem 2. Since $\frac{63n+50}{3}$ is not an integer, it follows that $\bar{c_3}(n) = c_3(n)$, and this completes the proof of the theorem.

□

## 4. Congruences for $c_3(n) \mod 5$

In this section we prove a congruence which is analogous to Euler's pentagonal number theorem.

**Main Theorem 2.** *Using the notation established above we find that*
$$c_3(5n+2) \equiv p\left(\frac{5n+2}{3}\right) \mod 5$$
*except when $n = 3T_m$ for some triangular number $T_m$. In these cases we find that*
$$c_3(15T_m+2) \equiv (-1)^m(m+3) \mod 5.$$

*Proof.* Consider the modular function $F(z)$ defined by
$$F(z) := \sum_{n\ge -1} a(n)q^n = \frac{\eta^3(9z)}{\eta^3(z)\eta(3z)}\frac{\eta^5(z)}{\eta(5z)}\frac{1}{\eta^3(15z)}.$$

The first few terms of $F(z)$ are
$$F(z) = \sum_{n\ge -1} a(n)q^n = q^{-1} - 2 - q + 3q^2 - q^3 + 2q^4 - 4q^6 + \dots.$$

We claim that $a(5n) \equiv 0 \mod 5$ for all $n \ge 1$. Notice that $a(0) = -2$.

Now construct a weight 12 cusp form $F_2(z)$ by multiplying $F(z)$ by $\eta^{24}(5z)$. In particular we have
$$F_2(z) := \eta^{24}(5z)F(z) = \sum_{n\ge 1}\tau(n)q^{5n}\sum_{n\ge -1}a(n)q^n = \sum_{n\ge 4}c(n)q^n.$$

Here $\tau(n)$ is Ramanujan's function. By Theorem 1 and (11) we find that $F_2(z) \in S_{12}(180, Id)$ where $Id$ is the trivial Dirichlet character $\mod 180$. It is clear that

$$\begin{aligned} c(5n) &= \sum_{k=0}^{n-1} a(5k)\tau(n-k) \\ &\equiv 3\tau(n) + a(5(n-1)) + \sum_{k=1}^{n-2} a(5k)\tau(n-k) \mod 5. \end{aligned}$$

Moreover if $c(5n) \equiv 3\tau(n) \mod 5$ for all $n \geq 1$, then by induction this implies that $a(5n) \equiv 0 \mod 5$ for all $n \geq 1$ by induction. Now we only need to compare $T_5|F_2(z) = \sum_{n\geq 1} c(5n)q^n$ and $\eta^{24}(z) = \sum_{n\geq 1} \tau(n)q^n \mod 5$. By Sturm's theorem, we only need to check that the congruence holds for the first $5 \cdot \frac{12}{12} \cdot [SL_2(\mathbb{Z}) : \Gamma_0(180)] + 1 = 2161$ terms which is easily verified.

Therefore we find that $a(5n) \equiv 0 \mod 5$ for all $n \geq 1$. We now note that

$$F(z) \prod_{n=1}^{\infty} (1 - q^{15n})^3 \equiv q^{-2} \sum_{n=0} \bar{c_3}(n) q^n \mod 5.$$

By Jacobi's Triple product formula, we know that

$$\prod_{n=1}^{\infty} (1 - q^{15n})^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{15T_m}.$$

Since $a(5n) \equiv 0 \mod 5$ for all $n \geq 1$, we obtain

$$\bar{c_3}(5n+2) \equiv 0 \mod 5$$

except when $5n + 2 = 15T_m + 2$ in which case

$$\bar{c_3}(15T_m + 2) \equiv (-1)^m (2m+1) a(0) = (-1)^m (m+3) \mod 5.$$

Since $\frac{15T_m+2}{3}$ is not an integer, $\bar{c_3}(15T_m+2) = c_3(15T_m+2)$, we obtain the claimed result.

□

As an immediate corollary we obtain

**Corollary.** *If $n \not\equiv 2 \mod 5$ and $n \neq 3T_m$ for any integer $m$, then*

$$c_3(5n+2) \equiv 0 \mod 5.$$

## 5. Acknowledgements

I am indebted to Kevin James whose computations verified the congruences in this paper.

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