CONGRUENCES FOR PARTITION FUNCTIONS

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Dedicated to Professor Heini Halberstam on the occasion of his retirement

ABSTRACT. The ordinary partition function \( p(n) \) and some of its generalizations satisfy some beautiful congruence properties. For instance, Ramanujan proved that for every integer \( n \)

\[
\begin{align*}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}
\]

Here we consider congruences for \( p(n) \) and \( c_h(n) \), the number of partitions of \( n \) into \( h \) colors. If \( l \) is prime and \( s \) is a positive integer, then, using a result of Sturm, we compute a constant \( C(h, t, r; l^s) \) such that \( c_h(tn + r) \equiv 0 \pmod{l^s} \) for all \( n \) if and only if the congruence holds for every \( n \leq C(h, t, r; l^s) \). If \( h = 1 \), these results pertain to \( p(n) \). In many cases, \( C(h, t, r; l^s) \) is small enough that one obtains an effective method of determining the truth of alleged congruences. For example, Ramanujan's congruences are easily verified because \( C(1, 5, 4, 5) = 2, C(1, 7, 5, 7) = 4, \) and \( C(1, 11, 6, 11) = 10. \)

1. PRELIMINARIES

An ordinary partition of a non-negative integer \( n \) is a non-increasing sequence of positive integers with sum \( n \). Let \( p(n) \) denote the ordinary partition function; that is, let \( p(n) \) denote the number of partitions of \( n \). Ordinary partitions have many interesting generalizations, including partitions of \( n \) into \( h \) colors. A partition of a non-negative integer \( n \) into \( h \) colors is a non-increasing sequence of positive integers in which each positive integer is assigned one of \( h \) distinct colors; the order of the colors is not considered, and the sum of all of these positive integers is \( n \). If \( c_h(n) \) denotes the number of partitions of \( n \) into \( h \) colors, then it is easy to see that the generating function for \( c_h(n) \) is:

\[
\sum_{n=0}^{\infty} c_h(n) q^n = \prod_{n=1}^{\infty} \left( \frac{1}{1-q^n} \right)^h.
\]

Of course, if \( h = 1 \), then \( c_1(n) = p(n) \).

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These partition functions possess a number of outstanding congruence properties. The most well known examples of these congruences are the Ramanujan Congruences. Some examples of colored partition congruences were given by Gandhi [4], who exhibited infinite families of congruences using identities of Euler and Jacobi. For instance, he proved that
\[
c_2(5n + 3) \equiv 0 \pmod{5},
\]
\[
c_8(11n + 4) \equiv 0 \pmod{11}.
\]

Other colored partition congruences were also found by Newman [13].

There are a number of methods that prove such congruences. Often these proofs depend on the manipulation of generating functions, the theory of modular forms, or beautiful combinatorial arguments where the partitions are equally distributed into various equivalence classes (see [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 16, 18]). The purpose of this paper is to define a simple algorithm for which the truth of an alleged congruence for \( c_k(n) \) is easily verified by computing enough values.

We shall use the theory of modular forms to obtain our results. We begin by recalling some essential preliminaries. If \( N \) is a positive integer, then \( \Gamma_0(N) \) and \( \Gamma_1(N) \) denote subgroups of \( SL_2(\mathbb{Z}) \) that are defined by:

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}
\]

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \ a \equiv d \equiv 1 \pmod{N}, \text{ and } c \equiv 0 \pmod{N} \right\}.
\]

It is well known [11, p.107] that
\[
[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p | N} \left( 1 + \frac{1}{p} \right)
\]
and
\[
[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p | N} \left( 1 - \frac{1}{p^2} \right).
\]

Let \( SL_2(\mathbb{Z}) \) act on \( \mathfrak{H} \), the upper half of the complex plane, in the usual way. That is, if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), then \( Az = \frac{az + b}{cz + d} \). If \( \chi \) is a Dirichlet character modulo \( N \) and \( k \) is a positive integer, then any meromorphic function \( f(z) \) on \( \mathfrak{H} \) satisfying
\[
f(Az) = \chi(d)(cz + d)^k f(z)
\]
for all \( A \in \Gamma_0(N) \), \( z \in \mathfrak{H} \) is called a modular form of weight \( k \) and Nebentypus character \( \chi \) with respect to \( \Gamma_0(N) \). If \( f(z) \) is holomorphic on \( \mathfrak{H} \) and at the cusps (that is, the rationals) of \( \Gamma_0(N) \), then \( f(z) \) is called a holomorphic modular form of type \( (k, \chi) \) on \( \Gamma_0(N) \). If \( f(z) \) is such a form that also vanishes at the cusps, then it is called a cusp form of type \( (k, \chi) \) on \( \Gamma_0(N) \).

The set of weight \( k \) holomorphic modular (resp. cusp) forms of type \( (k, \chi) \) on \( \Gamma_0(N) \) forms a finite dimensional vector space over \( \mathbb{C} \) which we denote by \( M_k(N, \chi) \) (resp. \( S_k(N, \chi) \)).
We shall also require other modular forms. If \( f(z) \) is a meromorphic function on \( \mathcal{H} \) and
\[
f(Az) = (cz + d)^k f(z)
\]
for all \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \), then \( f(z) \) is called a modular form of weight \( k \) with respect to \( \Gamma_1(N) \). If \( f(z) \) is holomorphic on \( \mathcal{H} \) and at the cusps of \( \Gamma_1(N) \), then \( f(z) \) is called a holomorphic modular form with respect to \( \Gamma_1(N) \). Moreover, if \( f(z) \) vanishes at the cusps, then \( f(z) \) is called a cusp form of weight \( k \) with respect to \( \Gamma_1(N) \). The set of holomorphic modular (resp. cusp) forms of weight \( k \) forms a finite dimensional vector space over \( \mathbb{C} \) which we denote by \( M_k(N) \) (resp. \( S_k(N) \)). It is easy to see that if \( f(z) \in M_k(N, \chi) \) (resp. \( S_k(N, \chi) \)) for any \( \chi \), then \( f(z) \in M_k(N) \) (resp. \( S_k(N) \)).

If \( f(z) \) is a holomorphic modular form with respect to \( \Gamma_1(N) \), then it admits a Fourier expansion in the variable \( q = e^{2\pi i z} \) of the form
\[
f(z) = \sum_{n=0}^{\infty} a(n)q^n.
\]
A crucial function in the study of modular forms is Dedekind's eta-function. It is defined by
\[
\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]
It is important to note that \( \eta(z) \) is a weight \( \frac{1}{2} \) holomorphic modular form that is non-vanishing on \( \mathcal{H} \). From this, we now see that the generating function for \( p(n) \) may be written as \( q^{\frac{1}{24}} \eta^{-1}(z) \), and that the generating function for \( a_k(n) \) may be written as \( q^{\frac{k}{24}} \eta^{-h}(z) \).

By [8], it is known that if \( f(z) = \prod_{d|N} \eta^{r_d}(\delta z) \) is an eta-product where
\[
(1) \quad \sum_{d|N} r_d \equiv 0 \pmod{24}
\]
and
\[
(2) \quad N \sum_{d|N} \frac{r_d}{d} \equiv 0 \pmod{24},
\]
then \( f(z) \) is a modular form with respect to \( \Gamma_0(N) \) with Nebentypus character \( \chi \) where \( k = \frac{1}{2} \sum_{d|N} r_d \) and \( \chi \) is defined by \( \chi(d) = \left(\frac{-1}{d}\right)^{w} \left(\frac{N}{d}\right)^{\chi^d} \) where \( w = \prod_{d|N} \delta^s \). Note that when \( \chi \) is defined in this manner, it is a Dirichlet character modulo \( N \). Notice that if \( k \) is a positive integer and \( f(z) \) is holomorphic at all of the cusps, then \( f(z) \in M_k(N, \chi) \).

Fortunately, it is fairly easy to deduce whether or not \( f(z) \) is holomorphic at the cusps of \( \Gamma_0(N) \). Since every cusp of \( \Gamma_0(N) \) has a representative of the form \( \frac{N}{d} \) where \( d \mid N, c \) and \( d \) are positive integers, and \( \gcd(c, d) = 1 \), it suffices to know the orders of \( f(z) \) at all such cusps. By [12], the order of \( f(z) \) at such a cusp \( \frac{N}{d} \) is
\[
(3) \quad \frac{N}{24d \cdot \gcd(d, \frac{N}{d})} \sum_{d|N} \frac{\gcd(d, \delta)^2 r_d}{\delta}.
\]
Now we recall the Hecke operators $T_m$. The Hecke operators are linear transformations on the vector space $M_k(N, \chi)$ that preserve both $M_k(N, \chi)$ and $S_k(N, \chi)$. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(N, \chi)$, then the modular form $f(z) \mid T_m \in M_k(N, \chi)$ is defined by

$$f(z) \mid T_m = \sum_{n=0}^{\infty} \sum_{d \mid \gcd(m,n)} \chi(d) d^{k-1} a(nm/d^2)q^n.$$  

In this paper, it is important to know when two holomorphic modular forms $f(z)$ and $g(z)$ are congruent modulo a positive integer $j$. Let $j$ be a positive integer and let $F(q) = \sum_{n \geq N_0} A(n)q^n$ be a formal power series with integer coefficients. Then the function $\text{Ord}_j(F(q))$ is defined by

$$\text{Ord}_j(F(q)) := \min \{ n \mid A(n) \not\equiv 0 \pmod{j} \}.$$

A theorem of Sturm [17] says that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=0}^{\infty} b(n)q^n$ are holomorphic modular forms of weight $k$ with respect to some congruence subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ with integer coefficients, then $f(z) \equiv g(z) \pmod{l}$ where $l$ is prime if and only if

$$\text{Ord}_l(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma].$$

Sturm’s theorem can be easily extended from $l$ to an arbitrary integer $j$. Without loss of generality, we may assume that $j$ is a power of a prime, say $j = l^s$. Suppose that $f(z) \equiv g(z) \pmod{l^s}$ where $s > 1$. Let $F_1(z) := f(z) - g(z)$ and let $G_1(z) := 0$. Since both $F_1(z)$ and $G_1(z)$ are modular forms with integer coefficients, Sturm’s criterion allows us to deduce that $F_1(z) \equiv G_1(z) \pmod{l}$. Then let $F_2(z) := \frac{1}{l}F_1(z)$ and $G_2(z) := 0$. Since both $F_2(z)$ and $G_2(z)$ are weight $k$ modular forms with integer coefficients, Sturm’s criterion again allows us to deduce that $F_2(z) \equiv G_2(z) \pmod{l}$ which is equivalent to the statement that $f(z) \equiv g(z) \pmod{l^s}$. By iterating this process, it is easy to see that Sturm’s criterion holds in generality.

It will be convenient to use certain modular forms whose Fourier expansions are congruent to the constant 1 modulo powers of any given fixed prime. We shall use the following eta-product.

**Lemma 1.** If $l$ is a prime and $s \geq 1$, then the eta-product $\frac{\eta^{l^s}(z)}{\eta^{l^{s-1}}(l^s z)}$ satisfies

$$\frac{\eta^{l^s}(z)}{\eta^{l^{s-1}}(l^s z)} \equiv 1 \pmod{l^s}.$$  

**Proof.** We first note that if $f(q) = 1 + \sum_{n=1}^{\infty} a(n)q^n$ is a power series with integer coefficients such that $a(n) \equiv 0 \pmod{l}$ for all $n \geq 1$, then $f^{l^s}(q)$, the $l^s$ power of $f(q)$, satisfies

$$f^{l^s}(q) \equiv 1 \pmod{l^{s+1}}.$$  

This holds by hypothesis for $s = 0$, and if $f^{l^s}(q) = 1 + l^{s+1}g(q)$, then

$$f^{l^{s+1}}(q) = [1 + l^{s+1}g(q)]^l = 1 + l^{s+2}h(q),$$

completing the induction.

Since $1 - X^l \equiv (1 - X)^l \pmod{l}$, it is easy to see that $f(q) := \frac{\eta^{l^s}(z)}{\eta(l^s z)} \equiv 1 \pmod{l}$.  

2. THE MAIN RESULTS

In this section, we compute $C(h, t, r, l^g)$. The strategy is to construct a holomorphic integer weight modular form whose Fourier coefficients are all multiples of $l^g$ if and only if $a_n(n + r) \equiv 0 \pmod{l^g}$ for every integer $n$. Once such a form is constructed, Sturm's theorem provides the explicit constant $C(h, t, r, l^g)$.

The first two theorems cover all possible cases; however, to achieve such generality we often obtain constants that are far from optimal. The other theorems in this section illustrate cases where the results of the first two theorems are sharpened dramatically. Theorem 1 gives us a result if $l \geq 5$ is prime.

**Theorem 1.** Let $l \geq 5$ be prime and let $0 \leq r < t$. Then we have

$$c_n(n + r) \equiv 0 \pmod{l^g}$$

for every integer $n$ if and only if the congruence holds for every $n \leq C(h, t, r, l^g)$. The constant $C(h, t, r, l^g)$ is defined as follows:

1. Let
   $$m := \begin{cases} 
   8, & \text{if } h \equiv 1 \pmod{2} \text{ and } t \equiv 0 \pmod{2}; \\
   1, & \text{otherwise}. 
   \end{cases}$$

2. Let $b$ be any integer such that $b \geq \frac{ht}{\gcd(l, t)}$ and $b \geq t\left[h - \frac{1}{24}l^{g-2}(l^2 - 1)\right]$ for which $b \equiv h \pmod{2}$ and $mb \equiv mht \pmod{24}$.

Then define $C(h, t, r, l^g)$ by

$$C(h, t, r, l^g) := \frac{2^{12}3^45^213\ell\lim[t, b][l^{g-1}(l - 1) + b - h]}{\gcd(24t, 24r + bt - h)^2} \prod_{p \nmid m} \left(1 - \frac{1}{p^2}\right).$$

**Proof.** Let

$$f(z) = \frac{\eta^2(24t z)}{\eta^4(24z)} \left(\frac{\eta^4(z)}{\eta^2(z)}\right)^{l^{g-1}} = \sum_{n=0}^{\infty} a(n)q^n.$$

Using the notation from our discussion prior to the statement of Theorem 1, we see that $r_1 = l^g, r_2 = -l^{g-1}, r_3 = -h$, and $r_4 = b$. We can see that conditions (1) and (2) hold where $N = 24m \cdot \lim[t, l]$, and hence $f(z)$ is a modular form with respect to $\Gamma_1(N)$.

We would like to show that the orders at all of the cusps are non-negative. Since the scalar in front of the sum in (3) is always positive, it suffices to show that

$$l^g - \frac{l^{g-1}\gcd(d, l)^2}{l} - \frac{h \cdot \gcd(d, 24)^2}{24} + \frac{b \cdot \gcd(d, 24t)^2}{24t} \geq 0.$$

If $\gcd(d, l) = l$, we have that $l^g - \frac{l^{g-1}\gcd(d, l)^2}{l} = 0$, and so it suffices to show that $\frac{b \cdot \gcd(d, 24t)^2}{24t} \geq \frac{h \cdot \gcd(d, 24)^2}{24}$. This is equivalent to $b \geq \frac{ht \cdot \gcd(d, 24)^2}{\gcd(d, 24t)^2}$, but since $l \geq 5$
is prime and $l|d$, we have that 
\[ \frac{ht}{\gcd(l,t)^2} \geq \frac{ht \cdot \gcd(d,24)^2}{\gcd(d,24t)^2}, \]
and hence our inequality holds by our selection of $b$ in 2.

If $\gcd(d,l) = 1$, it suffices to show that 
\[ \eta(l) \cdot \gcd(d,24t)^2 \geq t[l \cdot \gcd(d,24)^2 - 24l^{2-2} (l^2 - 1)], \]
but since $\gcd(d,24t) \geq \gcd(d,24)$ and since $b \geq ht$ (see 2) the result follows.

Hence $f(z)$ is holomorphic at the cusps, and since $\eta(z)$ is non-vanishing and holomorphic on $\Gamma_0$, $f(z) \in M_k(N)$ where $k = \frac{1}{2}[l^{2-1}(l-1) + b - h]$. There are other holomorphic modular forms that could be used in place of $f(z)$, but we have defined $f(z)$ as above to efficiently cancel poles while keeping the level and the weight of the modular form to a minimum in an effort to achieve the best possible bound.

By Lemma 1, if $\eta(24tz) = \sum_{n=0}^{\infty} d(24tn)q^{24tn+bt}$, then
\[
(5) \quad a(24tm + 24r + bt - h) \equiv \sum_{m=0}^{\infty} d(24tm)c_h(tn + r - tm) \pmod{l^s}.
\]

Since $d(0) = 1$, (5) becomes
\[
a(24tn + 24r + bt - h) \equiv c_h(tn + r) + \sum_{m=1}^{\infty} d(24tm)c_h(tn + r - tm) \pmod{l^s}.
\]

By induction, it is easy to see that $c_h(tn + r) \equiv 0 \pmod{l^s}$ for all $n$ if $a(24tn + 24r + bt - h) \equiv 0 \pmod{l^s}$ for all $n$. Hence we have that $c_h(tn + r) \equiv 0 \pmod{l^s}$ for all $n$ if and only if $a(24tn + 24r + bt - h) \equiv 0 \pmod{l^s}$ for all $n$.

Now define
\[ f_1(z) := \sum_{n=24r + bt - h \mod 24t} a(n)q^n. \]

By [Lemma 2, 14, 15], it is known that $f_1(z) \in M_k(\frac{276N^2}{9})$ where $g = \gcd(24t,24r + bt - h)$. Hence, by Sturm's theorem, we find that $f_1(z) \equiv 0 \pmod{l^s}$ if and only if $a(24tn + 24r + bt - h) \equiv 0 \pmod{l^s}$ for all $24tn + 24r + bt - h \leq \frac{276N^2}{9} \prod_{p \mid 276t} \left(1 - \frac{1}{p}\right)$. Therefore $c_h(tn + r) \equiv 0 \pmod{l^s}$ for every integer $n$ if and only if the congruence holds for every $n \leq C(h,t,r,l^s)$.

The next theorem is the analog of Theorem 1 where $l = 2$ or 3.

**Theorem 2.** Let $l = 2$ or 3 and let $0 \leq r < t$. Then we have
\[ c_h(tn + r) \equiv 0 \pmod{l^s} \]
for every integer $n$ if and only if the congruence holds for every $n \leq C(h,t,r,l^s)$. The constant $C(h,t,r,l^s)$ is defined as follows:

1. Let
   \[ m := \begin{cases} 8, & \text{if } h \equiv 1 \pmod{2} \text{ and } t \equiv 0 \pmod{2}; \\ 1, & \text{otherwise}. \end{cases} \]

2. Let $b$ be any integer such that $b \geq ht$ for which $b \equiv h \pmod{2}$ and $mb \equiv mht \pmod{24}$. 

\[ \sum_{m=0}^{\infty} d(24tm)c_h(tn + r - tm) \equiv \sum_{m=0}^{\infty} d(24tm)c_h(tn + r - tm) \pmod{l^s}. \]
3. Let
\[ N := \begin{cases} 
24mt, & \text{if } s \geq 2; \\
48mt, & \text{if } s = 1 \text{ and } l = 2; \\
72mt, & \text{if } s = 1 \text{ and } l = 3. 
\end{cases} \]

Then define \( C(h,t,r,l^s) \) by
\[
C(h,t,r,l^s) := \frac{24N^2\beta^{l^s-1}(l-1) + b - h}{\text{gcd}(24t, 24r + br - h)^2} \prod_{p|N} \left( 1 - \frac{1}{p^3} \right).
\]

The proof of Theorem 2 is similar to the proof of Theorem 1.

Now, using Theorem 1 and Theorem 2, we can verify any alleged congruence for \( \chi_0(n) \) in any arithmetic progression. However, in many instances the bounds established by Theorem 1 and Theorem 2 are not optimal. Next we consider convenient cases which appear to cover most non-trivial congruences of this type. In these cases we can construct a modular form with trivial Nebentypus character with respect to \( \Gamma_0(N) \).

**Theorem 3.** Let \( l \geq 5 \) be prime, let \( 0 \leq r < t \), and suppose \( h \) is odd. If \( t \) and \( h \) satisfy either
- \( \text{gcd}(t,6) = 1 \)
- \( \text{gcd}(t,6) = 3 \) and \( h \equiv 0 \mod 3 \),

then define \( N \) and \( b \) by:

1. Let \( N \) be the smallest integer multiple of \( \text{lcm}[t,l] \) such that \( N/l \) is a perfect square.
2. Let \( b \) be any integer such that \( b \geq hN/l^2 \) and \( b \geq N[h - l^{s-2}(l^2 - 1)] \) for which \( b \equiv hN \mod (24) \).

If \( r \equiv \frac{-hN+b}{24} \mod t \), then
\[ \chi_0(tn+r) \equiv 0 \mod l^s \]
for every integer \( n \) if and only if the congruence holds for every \( n \leq C(h,t,r,l^s) \). Here \( C(h,t,r,l^s) \) is defined by
\[
C(h,t,r,l^s) := \frac{N[l^{s-1}(l-1) + b - h]}{24} \prod_{p|N} \left( 1 + \frac{1}{p} \right).
\]

**Proof.** Let
\[
f(z) = \frac{\eta^l(Nz)}{\eta^l(z)} \left( \frac{\eta^l(z)}{\eta(z)} \right)^{l^s-1} = \sum_{n=0}^{\infty} a(n)q^n.
\]

Using the notation from our discussion prior to the statement of Theorem 1, we see that \( r_1 = l^s - h, r_1 = -l^{s-1}, \) and \( r_N = b \). We can see that conditions (1) and (2) hold where \( N \) is defined as above. We can also see that \( w = \prod_{p|N} \delta_{p^s} \) is a square and that \( k = \frac{1}{2}[l^{s-1}(l-1) + b - h] \) is even, and hence \( f(z) \) is a modular form of weight \( k \) and character \( \chi_0 \), the trivial character, with respect to \( \Gamma_0(N) \).

We would like to show that the orders at all of the cusps are non-negative. Since the scalar in front of the sum in (3) is always positive, it suffices to show that
\[
\frac{l^s}{1} - \frac{l^{s-1}\text{gcd}(d,N)^2}{l} - \frac{h}{1} + \frac{b \cdot \text{gcd}(d,N)^2}{N} \geq 0.
\]
If \( \gcd(d,l) = l \), we have that
\[
\frac{l^6}{l} - \frac{l^{s-1}\gcd(d,l)^2}{l} = 0,
\]
and so it suffices to show that
\[
\frac{b \cdot \gcd(d,N)^2}{N} \geq h.
\]
This is equivalent to \( b \geq \frac{hN}{\gcd(d,N)^2} \), but since \( \ell | d \) and \( \ell | N \), we have that
\[
\frac{hN}{\ell} \geq \frac{hN}{\gcd(d,N)^2},
\]
and hence our inequality holds by our selection of \( b \) above.

If \( \gcd(d,l) = 1 \), it suffices to show that
\[
b \cdot \gcd(d,N)^2 \geq N[h - l^{s-2}(l^2 - 1)],
\]
but since \( \gcd(d,N) \geq 1 \) our inequality again holds by our selection of \( b \) above. Hence \( f(z) \) is holomorphic at the cusps, and since \( \eta(z) \) is non-vanishing and holomorphic on \( \mathcal{H} \), \( f(z) \in M_k(N,\chi_0) \). There are other holomorphic modular forms that could be used in place of \( f(z) \), but we have defined \( f(z) \) as above to efficiently cancel poles while keeping the level and the weight of the modular form to a minimum in an effort to achieve the best possible bound.

By Lemma 1, we may conclude that if \( \eta^N f(Nz) = \sum_{n=0}^{\infty} d(Nn)q^{Nn} \), then
\[
a(tn + r + \frac{bN - h}{24}) \equiv \sum_{m=0}^{\infty} d(Nm)\eta_n(tn + r - Nm) \pmod{l^6}.
\]

Since \( d(0) = 1 \), (6) becomes
\[
a(tn + r + \frac{bN - h}{24}) \equiv c_n(tn + r) + \sum_{m=1}^{\infty} d(Nm)\eta_n(tn + r - Nm) \pmod{l^6}.
\]

By induction, it is easy to see that \( c_n(tn + r) \equiv 0 \pmod{l^6} \) for all \( n \) if \( a(tn + r + \frac{bN - h}{24}) \equiv 0 \pmod{l^6} \) for all \( n \). Hence we have that \( c_n(tn + r) \equiv 0 \pmod{l^6} \) for all \( n \) if and only if \( a(tn + r + \frac{bN - h}{24}) \equiv 0 \pmod{l^6} \) for all \( n \).

Now notice that \( r + \frac{bN - h}{24} \equiv 0 \pmod{t} \) by hypothesis, so let us consider
\[
f_1(z) = f(z)\eta_t = \sum_{n=0}^{\infty} a(tn)q^n.
\]
Notice that \( f_1(z) \in M_k(N,\chi_0) \). Hence, by Sturm's theorem, we find that \( f_1(z) \equiv 0 \pmod{l^6} \) if and only if \( a(tn) \equiv 0 \pmod{l^6} \) for all \( n \) if and only if \( c_n(tn + r) \equiv 0 \pmod{l^6} \) for every integer \( n \) if and only if the congruence holds for every \( n \leq C(h, t, r, l^6) \).

The next theorem is the analog of Theorem 3 where \( l = 2 \) or 3.

**Theorem 4.** Let \( l = 2 \) or 3, let \( 0 \leq r < t \), and suppose \( h \) is odd. If \( t \) and \( h \) satisfy either
\[
\begin{align*}
&\gcd(t, 6) = 1, \\
&\gcd(t, 6) = 3 \text{ and } h \equiv 0 \pmod{3},
\end{align*}
\]
then define \( J \), \( N \), and \( b \) by:
\[
\begin{enumerate}
\item Let \( J \) be the smallest integer square such that \( t | J \).
\item Let \( b \) be any integer such that \( b \geq hJ / \gcd(l, t)^2 \) and \( b \geq J[h - 2l^{s-2}(l^2 - 1)] \) for which \( b \equiv hJ \pmod{24} \).
\end{enumerate}
\]
3. Let
\[ N := \begin{cases} 
J_s & \text{if } s \geq 4; \\
J_l & \text{if } s = 3 \text{ and } l = 3; \\
2^{4-s}J_s & \text{if } s < 4 \text{ and } l = 2; \\
3^{3-s}J_s & \text{if } s < 3 \text{ and } l = 3.
\end{cases} \]

If \( r \equiv -\frac{bl-b}{24} \pmod{t} \), then
\[ c_h(tn+r) \equiv 0 \pmod{f^t} \]

for every integer \( n \) if and only if the congruence holds for every \( n \leq C(h,t,r,f^t) \). Here \( C(h,t,r,f^t) \) is defined by
\[ C(h,t,r,f^t) := \frac{N[2^{4-s}(l-1) + b - b]}{24} \prod_{p|N} \left( 1 + \frac{1}{p} \right). \]

The proof of Theorem 4 is similar to the proof of Theorem 3 with
\[ f(z) = \frac{\eta_h(Jz)}{\eta_h(z)} \left( \frac{\eta(z)}{\eta(Jz)} \right)^{-2^{s-1}} \]

being the appropriate modular form.

By carefully selecting some of the techniques introduced in the proof of Theorem 3, one can prove theorems that are more general than Theorems 3 and 4 which produce bounds smaller than those in Theorems 1 and 2.

3. Examples

Now we use these theorems to obtain new proofs of some well known congruences.

**Corollary 1.** If \( p(n) \) denotes the number of partitions of \( n \), then
\[ p(5n+4) \equiv 0 \pmod{5}, \]
\[ p(7n+5) \equiv 0 \pmod{7}, \]
\[ p(11n+6) \equiv 0 \pmod{11} \]

for all \( n \).

**Proof.** For the first congruence, we find that the hypotheses of Theorem 3 are met with \( J = b = l = 5 \), and hence \( C(1,5,4,5) = 2 \). Since \( p(5n+4) \equiv 0 \pmod{5} \) for \( 0 \leq n \leq 2 \), we find that \( p(5n+4) \equiv 0 \pmod{5} \) for all non-negative integers \( n \).

For the other two congruences, the hypotheses are met with \( J = b = l = 7 \) and \( J = b = l = 11 \) respectively. In these cases we find that \( C(1,7,5,7) = 4 \) and \( C(1,11,6,11) = 10 \).
Corollary 2. For all $n$,
\[ c_2(5n + 3) \equiv 0 \pmod{5} \]
and
\[ c_6(11n + 4) \equiv 0 \pmod{11}. \]

Proof. For the first congruence, we find that the hypotheses of Theorem 3 are met with $J = l = 5$ and $b = 10$, and hence $C(2, 5, 3, 5) = 3$. Since $c_2(5n + 3) \equiv 0 \pmod{5}$ for all $0 \leq n \leq 3$, we find that $c_2(5n + 3) \equiv 0 \pmod{5}$ for all non-negative integers $n$.

We find that the hypotheses of Theorem 3 are met with $J = l = 11$ and $b = 20$, and hence $C(8, 11, 4, 11) = 11$.

\[ \square \]

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