SOME RECURRENCES FOR ARITHMETICAL FUNCTIONS

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Journal of the Indian Mathematical Society, 62, 1996, pages 29-50.

Abstract. Euler proved the following recurrence for $p(n)$, the number of partitions of an integer n :

$$p(n)+\sum_{k=1}^{\infty}(-1)^k(p(n-\omega(k))+p(n-\omega(-k)))=0 \tag{1}$$

for $\omega(k)=\frac{3k^2+k}{2}$. Using the Jacobi Triple Product identity we show analogues of Euler's recurrence formula for common restricted partition functions. Moreover following Kolberg, these recurrences allow us to determine that these partition functions are both even and odd infinitely often. Using the theory of modular forms, these recurrences may be viewed as infinite product identities involving Dedekind's η-function. Specifically, if the generating function for an arithmetical function is a modular form, then one often obtains analogous recurrence formulas; in particular here we get recurrence relations involving the number of t-core partitions, the number of representations of sums of squares, certain divisor functions, the number of points in finite fields on certain elliptic curves with complex multiplication, the Ramanujan $\tau-$function and some appropriate analogs. In some cases recurrences hold for almost all n, and in others these recurrences hold for all n where the equality is replaced by a congruence $\pmod m$ for any fixed integer m. These new recurrences are consequences of some of the theory of modular forms as developed by Deligne, Ribet, Serre, and Swinnerton-Dyer.

1. Introduction

In the theory of partitions, one finds that there are many interesting properties which are exhibited by various partition functions. In particular one of the crowning achievements is the Hardy-Ramanujan-Rademacher asymptotic formula for the number of partitions of n, which we denote by $p(n)$,

$$p(n)\sim\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}$$

Key words and phrases. Recurrence relations, partitions, modular forms.

The congruential behavior of $p(n)$ also has been of significant interest. The most fascinating of the congruence properties of $p(n)$ are the special cases of the Ramanujan congruences:

$$p(5n+4) \equiv 0 \mod 5,$$

$$p(7n+5) \equiv 0 \mod 7,$$

and

$$p(11n+6) \equiv 0 \mod 11.$$

The behavior of $p(n) \mod 2$ however has been mystifying; computational evidence suggests that we expect $\sim \frac{1}{2}x$ many $n \leq x$ where $p(n)$ is even. Little is known in the direction of this conjecture. However in [8], Kolberg proves that the partition function is both even and odd infinitely often using the recurrence (1). Here we apply the methods of Kolberg to show that other natural restricted partition functions are both odd and even infinitely often.

In the second section we derive recurrence formulas for restricted partition functions using classical techniques, namely the

Jacobi Triple Product Identity. *If x, z are complex numbers such that $|x| < 1$ and $z \neq 0$, then*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^{-1})(1+x^{2n-1}z) = \sum_{n=-\infty}^{\infty} x^{n^2}z^n = 1 + \sum_{n=1}^{\infty} x^{n^2}(z^n + z^{-n})$$

and

Euler's Pentagonal Number Formula. *For x a complex number such that $|x| < 1$,*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}} = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)} = 1 + \sum_{n=1}^{\infty}(-1)^n(x^{\omega(n)} + x^{\omega(-n)})$$

A very useful consequence of the Jacobi Triple Product Identity is the following infinite product identity:

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n^2+n}{2}}. \tag{2}$$

In particular, we derive recurrence relations for $q(n)$ the number of partitions into distinct parts, $q_O(n)$ the number of partitions into distinct and odd parts, $p_E(n)$ the number of partitions into an even number of parts, and $p_O(n)$ the number of partitions into an odd number of parts. Moreover if r is a positive integer then we let $b_r(n)$ denote the number of partitions of n none of whose parts is a multiple of r. Finally we consider the partition function $g(n)$ which inherits a nice recurrence; here $g(n)$ denotes the number of partitions of n into parts none of which is a multiple of 4 and none of which is twice another. To do this we recall the generating functions for these restricted partition functions:

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}} \tag{3}$$

$$\sum_{n=0}^{\infty} q_O(n)x^n = \prod_{n=1}^{\infty}(1+x^{2n-1}) \tag{4}$$

$$\sum_{n=0}^{\infty}(p_E(n)-p_O(n))x^n = \prod_{n=1}^{\infty}(1-x^{2n-1}) \tag{5}$$

which together with

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty}(p_E(n)+p_O(n))x^n = \prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

gives us generating functions for $p_E(n)$ and $p_O(n)$. Also

$$\sum_{n=0}^{\infty} b_r(n)x^n = \prod_{n=1}^{\infty}\frac{1-x^{rn}}{1-x^n} \tag{6}$$

and

$$\sum_{n=0}^{\infty} g(n)x^n = \prod_{n=1}^{\infty}(1+x^{2n-1}+x^{4n-2}) \tag{7}$$

In the third section we get recurrence relations involving t-core partitions, the number of representations as sums of squares, certain divisor functions, the number of points in finite fields of certain elliptic curves, and the Ramanujan τ-function.

Definition. A partition is a *t-core* partition if none of the hook numbers are multiples of t.

Example. *Consider the partition of $n=7$, $7=3+3+1$. We can represent this by the Ferrer's graph*

$$\begin{array}{cccc} & 1 & 2 & 3 \\ 1 & \bullet_{(1,1)} & \bullet_{(1,2)} & \bullet_{(1,3)} \\ 2 & \bullet_{(2,1)} & \bullet_{(2,2)} & \bullet_{(2,3)} \\ 3 & \bullet_{(3,1)} & & \end{array}$$

A hand is the rightmost node of any row; here the hands are $(1,3),(2,3)$ and $(3,1)$. A foot is the bottom node of any column; here the feet are $(3,1),(2,2)$ and $(2,3)$. If (i,j) is a hand and (k,l) is a foot such that $i\le k$ and $l\le j$, then we can define a *hook connecting them. The arm of the hook connecting (i,j) and (k,l) consists of all the nodes (i,s) where $l \leq s \leq j$. The leg of this hook consists of all nodes (t,l) where $i \leq t \leq k$. The hook number of this hook is the number of nodes on the hook; precisely the hook number for this hook is $j-l+k-i+1$. The hook numbers in the above Ferrers graph are 1, 2, 3, 4, and 5.*

The generating function for $c_t(n)$, the number of t-core partitions of n is

$$\sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{tn})^t}{(1-x^n)}$$

The t-cores have been useful in [4] where several Ramanujan congruences for $p(n)$ are proved. These partitions also arise in the theory of modular representations of symmetric groups [6,9].

Also of interest are the following generating functions for the number of representations of integers as sums of positive odd integer squares and squares:

$$\theta_O(z) = \sum_{n=1,odd}^{\infty} q^{n^2} = q\prod_{n=1}^{\infty} \frac{(1-q^{16n})^2}{1-q^{8n}} = q + q^9 + q^{25} + \ldots \tag{8}$$

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots . \tag{9}$$

¿From the definition it is obvious that the coefficient of q^m in $\theta_O^k(z)$ is the number of ways to represent m as a sum of k positive integral odd squares. In particular when $k=3$ we will get a recurrence for this representation number which holds for almost all m (i.e. for all m in a set of density one in the positive integers). For more on the theory of partitions see [1].

2. Recurrences from Jacobi's Triple Product Formula

Henceforth, let n be a positive integer. We now prove an analogue of Euler's recurrence for $q(n)$ the number of partitions of n into distinct parts.

Theorem 1. *For $q(n)$ the number of partitions of n into distinct parts,*

$$q(n) + \sum_{k=1}^{\infty} (-1)^k (q(n-2\omega(k)) + q(n-2\omega(-k))) = \begin{cases} 1 \text{ if } n = \frac{m(m+1)}{2} \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let $z = x$ in the Jacobi Triple Product Identity. We obtain

$$\prod_{n=1}^{\infty} (1-x^{2n})(1+x^{2n})(1+x^{2n-2}) = \sum_{n=-\infty}^{\infty} x^{n^2+n}$$

that is,

$$\prod_{n=1}^{\infty}(1-x^{4n})(1+x^{2n-2}) = \sum_{n=-\infty}^{\infty} x^{n^2+n}$$

Replacing x^2 by x, one has:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{n-1}) = \sum_{n=-\infty}^{\infty} x^{\frac{n^2+n}{2}}$$

But $\prod_{n=1}^{\infty}(1+x^{n-1}) = 2\prod_{n=1}^{\infty}(1+x^n)$ and $\sum_{n=-\infty}^{\infty} x^{\frac{n^2+n}{2}} = 2+2\sum_{n=1}^{\infty} x^{\frac{n^2+n}{2}}$ so that

$$\prod_{n=1}^{\infty}(1-x^{2n})\prod_{n=1}^{\infty}(1+x^n) = 1+\sum_{n=1}^{\infty} x^{\frac{n^2+n}{2}}.$$

But the left hand side is the product of the generating function of $q(n)$ (3) and Euler's Pentagonal Number Formula where x is replaced by x^2, so

$$(1+\sum_{k=1}^{\infty}(x^{2\omega(k)}+x^{2\omega(-k)})(-1)^k)(\sum_{n=0}^{\infty} q(n)x^n) = 1+\sum_{n=1}^{\infty} x^{\frac{n^2+n}{2}}$$

Comparing coefficients on x^n we get the desired result. □

A similar approach can be used for other restricted partition functions, in fact, for $q_O(n)$, the number of partitions of n with distinct odd parts, the recurrence requires even less manipulation.

Theorem 2. *For $q_O(n)$ the number of partitions of n into distinct odd parts,*

$$q_O(n)+\sum_{k=1}^{\infty}(-1)^k(q_O(n-\omega(k))+q_O(n-\omega(-k)) = \begin{cases} 2(-1)^m \text{ if } n=2m^2 \\ 0 \text{ otherwise }. \end{cases}.$$

Proof. Let $z=i$ in the Jacobi Triple Product. Then

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}i)(1-x^{2n-1}i) = 1+\sum_{n=1}^{\infty} x^{n^2}(i^n+(-i)^n)$$

so

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{4n-2}) = 1+\sum_{n=1}^{\infty} x^{n^2}(2\cos\frac{n\pi}{2}) = 1+\sum_{n=1}^{\infty} 2(-1)^n x^{4n^2}.$$

If we replace x^2 by x, we get

$$\prod_{n=1}^{\infty}(1-x^n)(1+x^{2n-1}) = 1+\sum_{n=1}^{\infty} 2(-1)^n x^{2n^2}.$$

So using the Pentagonal Number Formula and the generating function for $q_O(n)$ (4) we get

$$(1+\sum_{k=1}^{\infty}(-1)^k(x^{\omega(k)}+x^{\omega(-k)}))(\sum_{n=0}^{\infty} q_O(n)x^n) = 1+\sum_{n=1}^{\infty} 2(-1)^n x^{2n^2}.$$

Comparing coefficients of like powers of x we get the result. □

Similarly we can treat $p_E(n)$, the number of partition of n into an even number of parts using the Triple Product Identity specialized for z.

Theorem 3. *For $p_E(n)$ the number of partitions of n into an even number of parts,*

$$p_E(n)+\sum_{k=1}^{\infty}(-1)^k(p_E(n-\omega(k))+p_E(n-\omega(-k)))=\begin{cases}(-1)^m \text{ if } n=m^2\\ 0 \text{ otherwise.}\end{cases}.$$

Proof. Let $z=-1$ in Jacobi's Triple Product. Then we get

$$\prod_{n=1}^{\infty}(1-x^{2n})(1-x^{2n-1})^2=1+\sum_{n=1}^{\infty}2(-1)^n x^{n^2}.$$

Since $1-x^{2n}=(1-x^n)(1+x^n)$ and $\prod_{n=1}^{\infty}(1+x^n)=\prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}$, we get

$$\prod_{n=1}^{\infty}(1-x^{n})(1-x^{2n-1})=1+\sum_{n=1}^{\infty}2(-1)^n x^{n^2}.$$

But now the left hand side is just the generating function of $p_E(n)-p_O(n)$ (5) times the Pentagonal Number Formula, so

$$p_E(n)-p_O(n)+\sum_{n=1}^{\infty}(-1)^k(p_E(n-\omega(k))-p_O(n-\omega(k))+p_E(n-\omega(-k))-p_O(n-\omega(-k)))$$

$$=\begin{cases}2(-1)^m \text{ if } n=m^2\\ 0 \text{ otherwise.}\end{cases}.$$

Adding this to the Euler's recurrence (1) gives

$$2p_E(n)+2\sum_{k=1}^{\infty}(-1)^k(p_E(n-\omega(k))+p_E(n-\omega(-k)))=\begin{cases}2(-1)^m \text{ if } n=m^2\\ 0 \text{ otherwise.}\end{cases}$$

Dividing both sides by two we get the desired result. □

Given the connection among $p(n), p_E(n)$, and $p_O(n)$, we may use the recurrence formulas for the first two partition functions to get one for the third.

Theorem 4. *If $p_O(n)$ is the number of partitions of n into an odd number of parts,*

$$p_O(n)+\sum_{k=1}^{\infty}(-1)^k(p_O(n-\omega(k))+p_O(n-\omega(-k)))=\begin{cases}(-1)^{m-1} \text{ if } n=m^2\\ 0 \text{ otherwise.}\end{cases}$$

Proof. This follows from $p_E(n)+p_O(n)=p(n)$, Euler's recurrence (1), and the previous theorem. □

Again, by the generating function for $b_r(n)$ it is clear that multiplying by the Pentagonal Number Formula should give us a recurrence.

Theorem 5. *If r is a positive integer, then let $b_r(n)$ denote the number of partitions of n none of whose parts is a multiple of r. Then*

$$b_r(n)+\sum_{k=1}^{\infty}(-1)^k(b_r(n-\omega(k))+b_r(n-\omega(-k)))=\begin{cases}(-1)^m \text{ if } n=r\left(\frac{3m^2+m}{2}\right)\\ 0 \text{ otherwise.}\end{cases}$$

Proof. ¿From (6) we know that

$$\sum_{n=0}^{\infty} b_r(n)x^n=\prod_{n=1}^{\infty}\frac{1-x^{rn}}{1-x^n}$$

Therefore

$$\prod_{n=1}^{\infty}(1-x^n)\sum_{n=0}^{\infty} b_r(n)x^n=\prod_{n=1}^{\infty}(1-x^{rn})$$

which reduces to

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}\sum_{n=0}^{\infty} b_r(n)x^n=\sum_{n=-\infty}^{\infty}(-1)^n x^{r\omega(n)}$$

We now get the result by comparing coefficients of like powers of x. □

It might be noted that in our proofs of Theorems 2 and 3 we evaluated the Triple Product Identity at roots of unity. If we consider evaluating at $e^{\frac{2\pi i}{3}}$, a third root of unity, we get a more exotic recurrence.

Theorem 6. *If $g(n)$ is the number of partitions of n into distinct parts none of whose parts is a multiple of 4 and none of whose parts is twice another part, then*

$$g(n)+\sum_{k=1}^{\infty}(-1)^k(g(n-2\omega(k))+g(n-2\omega(-k)))=\begin{cases}2(-1)^m \text{ if } n=m^2, 3|m\\ (-1)^{m+1} \text{ if } n=m^2, 3\not|m\\ 0 \text{ otherwise.}\end{cases}$$

Proof. If we specialize the Jacobi Triple Product at $z=e^{\frac{2\pi i}{3}}$ then we get

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}e^{\frac{2\pi i}{3}})(1+x^{2n-1}e^{\frac{-2\pi i}{3}})=1+\sum_{n=1}^{\infty}x^{n^2}(e^{\frac{2\pi i n}{3}}+e^{\frac{-2\pi i n}{3}})$$

so we get

$$\prod_{n=1}^{\infty}(1-x^{2n})(1-x^{2n-1}+x^{4n-2})=1+\sum_{n=1}^{\infty}x^{n^2}(2\cos\frac{2\pi n}{3}).$$

Replacing x by $-x$ our left hand side then become Euler's Pentagonal Number Formula times the generating function for $g(n)$ (7),

$$(\sum_{n=0}^{\infty}g(n)x^n)(1+\sum_{k=1}^{\infty}(x^{2\omega(k)}+x^{2\omega(-k)})(-1)^k)=1+\sum_{3|n}2(-1)^n x^{n^2}+\sum_{3\not|n}(-1)^{n+1}x^{n^2}.$$

Comparing coefficients on like powers of x, we get the result. □

We can use these recurrence formulas to answer questions about the parity of these partition functions. Using the fact that $p(n)$ changes parity infinitely often [8] we can say the same thing about $q_O(n)$. This follows since the generating function for $q_O(n)$ is congruent to Euler's generating function for $p(n) \mod 2$ by the Children's Binomial Theorem. Precisely we find that

$$\begin{aligned}\sum_{n=0}^{\infty} q_O(n)x^n &= \prod_{n=1}^{\infty}(1+x^{2n+1}) \\ &\equiv \prod_{n=1}^{\infty}(1-x^{2n+1}) = \prod_{n=1}^{\infty}\frac{1-x^n}{1-x^{2n}} \\ &\equiv \prod_{n=1}^{\infty}\frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n \mod 2.\end{aligned}$$

We establish similar theorems for the other partition functions we have considered. To do so we give a preliminary

Lemma. *The following equations have no solutions a, m in the integers.*

$$\begin{aligned}\omega(a)+5 &= m^2 \\ \omega(a)+6 &= m^2 \\ 3a^2-a+3 &= m^2 \\ 3a^2-a+10 &= m^2.\end{aligned}$$

Proof. For the first two, consider getting $\omega(a)$ alone and then multiplying by 24 and adding 1. This allows us to complete the square, so we get

$$(6a+1)^2 - 24m^2 = -119$$

and

$$(6a+1)^2 - 24m^2 = -143.$$

Replacing $6a+1$ by x and reducing the first equation modulo 7 and the second modulo 11 we get

$$x^2 - 3m^2 \equiv 0 \mod 7$$

and

$$x^2 - 2m^2 \equiv 0 \mod 11.$$

Since 3 is not a square modulo 7 and 2 is not a square modulo 11, we get the result for the first two equations. For the third and fourth equations, multiplying by 12, adding 1, letting $x = 6a-1$, and reducing modulo 5 or 7 yields

$$x^2 - 2m^2 \equiv 0 \mod 5$$

and

$$x^2 - 5m^2 \equiv 0 \mod 7.$$

Again we get contradictions. □

These allow us to get results on the parity of our restricted partition functions.

Theorem 7. *Let $p_E(n)$ be the number of partitions of n into an even number of parts, $p_O(n)$ the number of partitions of n into an odd number of parts, and $g(n)$ the number of partitions of n into distinct parts, none of which is a multiple of four and none of which is twice another part. Then $p_E(n)$,$p_O(n)$, and $g(n)$ change parity infinitely often.*

Proof. Consider $p_E(n)$ and suppose it is even for all sufficiently large n, $n \geq a$. Since $p_E(7) = 7$, we have $a \geq 8$. By the recurrence for $p_E(n)$ Theorem 3 and the last lemma,

$$p_E(5+\omega(a)) + \ldots + (-1)^{a-1}(p_E(3a+4)+p_E(4a+3)) + (-1)^a(p_E(5+a)+p_E(5)) = 0.$$

Taking this equation modulo two we get a contradiction since $p_E(5) = 3$ is the only odd term on the left hand side of our equation. Now suppose $p_E(n)$ is odd for all $n \geq b$. Since $p_E(10) = 22$ we have $b \geq 11$. By Theorem 3 and the last lemma, we have

$$p_E(5+\omega(b)) + \ldots + (-1)^{b-1}(p_E(3b+4)+p_E(4b+3)) + (-1)^b(p_E(5)+p_E(5+b)) = 0.$$

Modulo two this leads to a contradiction since the left hand side has an odd number of odd terms. The result for $p_O(n)$ and $g(n)$ are treated similarly. □

3. Recurrences by the theory of modular forms

Whereas the method of proof in the previous section involved specializations of Jacobi's Triple Product Identity and multiplying by Euler's Pentagonal Number Formula, here we make use of the theory of modular forms, finding that there are many examples similar to those in the previous section as well as new forms of recurrences.

Let $SL_2(\mathbb{Z})$ be the group of 2×2 matrices with integer entries and determinant 1. Let $\mathfrak{H}$ be the upper half complex plane, i.e. the set of all complex numbers with positive imaginary part. $SL_2(\mathbb{Z})$ acts on the upper half complex plane naturally by the linear fractional transformation,

$$Az = \frac{az+b}{cz+d}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore any subgroup Γ of $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$. Of particular interest will be the *congruence subgroup* of level N, for N a positive integer:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$

Definition. *Let k and N be two positive integers and χ a Dirichlet character modulo N. Then a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a modular function of weight k with respect to $\Gamma_0(N)$ and χ if*

$$f(Az) = f(z)(cz+d)^k \chi(d)$$

for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

We also want to restrict the functions at the cusps, $\mathbb{Q} \cup \infty$ so that they don't have poles.

Definition. *A modular function is a modular form if it is holomorphic at every cusp.*

Throughout we let $q = e^{2\pi iz}$ and note that a holomorphic modular form $f(z)$ with respect to a congruence subgroup $\Gamma_0(N)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

An example of such a form is the Dedekind η-function,

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

It is a modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(576)$. For more on half-integral weight modular forms, see [14]. Another example, $\eta^{24}(z)$, is the famous Δ-function, a weight 12 form. If a form $f(z)$ is zero at all cusps, then we say that $f(z)$ is a *cusp form.*

There are also well known operators, the *Hecke operators*, $T(p)$ for each prime p whose action is given by

$$f(z)|T(p) = \sum_{n=0}^{\infty}(a(np) + \chi(p)p^{k-1}a(\frac{n}{p}))q^n$$

If $f(z)$ is a modular form of weight k with respect to $\Gamma_0(N)$ and character χ, then $f|T(p)$ is one as well. A modular form which is an eigenvector for each Hecke operator is called an *eigenform.* For more on the theory of modular forms the reader see [7,15].

A natural question to ask is what properties the Fourier coefficients of modular forms can have. There is a theorem of Serre [12] which says if the coefficients are algebraic integers then they have amazing divisibility properties.

Theorem(Serre). *If $f(z)$ is a modular form of positive integer weight k with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ where the $a(n)$ are algebraic integers in a fixed number field, then for any positive integer m,*

$$a(n) \equiv 0 \mod m$$

for almost all n (here we use almost all in the sense of on a set of density 1).

This theorem follows from multiplicativity of the coefficients of eigenforms and the theory of ℓ-adic Galois representations as developed by Deligne and Serre. This theorem tells us that whenever we can multiply the generating function of an arithmetic function by Euler's Pentagonal Number Formula or the Jacobi Triple Product Identity and obtain a modular form , then we get a congruential recurrence that almost always holds up to any modulus we choose. The philosophy then is that recurrences modulo any integer m that hold on a set of density 1 will be fairly plentiful. which hold $\mod m$ for any m should be fairly plentiful.

There are two types of forms that will allow us to do even better than just a recurrence up to modulus almost always. Consider $K = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field.

Let O_K be the ring of integers of K and fix an ideal Λ in O_K. Let $I(\Lambda)$ be the group of fractional ideals prime to Λ. A *Hecke Grossencharacter* of weight $k > 1$ is any homomorphism ϕ from $I(\Lambda)$ to $\mathbb{C}^\times$ such that

$$\phi(\alpha O_K) = \alpha^{k-1}$$

where $\alpha \equiv 1 \mod \Lambda$. Throughout we let $N(\mathfrak{a})$ denote the norm of ideal $\mathfrak{a}$. Hecke proved that the power series attached to a Grossencharacter is a cusp form of weight k. This motivates the definition of a modular forms with complex multiplication.

Definition. *If ϕ is a Hecke Grossencharacter of a quadratic imaginary field K with weight $k > 1$, then the $L-$series associated with ϕ is defined by:*

$$L(s) := \sum_{(\mathfrak{a},\Lambda)=1} \phi(\mathfrak{a}) N(\mathfrak{a})^{-s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where the sum is over all ideals in O_K that are relatively prime to the conductor Λ. The function $f(z)$ defined by the Mellin transform

$$f(z) := \sum_{n=1}^{\infty} a(n) q^n$$

is a cusp form of weight k with respect to the group $\Gamma_0(dN(\Lambda))$ where $N(\Lambda)$ is the ideal norm of Λ. This form $f(z)$ is called a modular form with complex multiplication (referred to as a CM form).

¿From the definition, it is clear that the Fourier coefficient $a(n)$ equals the sum of the values of ϕ over all ideals in O_K prime to Λ with norm n. For such forms half of the primes are inert so $a(p) = 0$ for half of the primes. By multiplicativity we see that $a(n) = 0$ for almost all n in a CM form and in a finite linear combination of CM forms. When a generating function for an arithmetic function times the Euler Pentagonal Number Formula is equal to a CM form, we have a recurrence that holds for almost all n. The Fourier expansion of a CM form is a special case of what is known as a lacunary power series. A series $\sum_{n=n_0}^{\infty} a(n)q^n$ is called *lacunary* if $a(n) = 0$ for almost all n. In this direction Serre has proven the following about the even powers of the Dedekind η-function [13].

Theorem(Serre). *The only positive integers r for which*

$$\prod_{n=1}^{\infty} (1 - q^n)^{2r}$$

is lacunary are $r = 1, 2, 3, 4, 5, 7, 13$.

Serre proved this theorem by applying a theorem of Ribet [11] which asserts that the Fourier expansion of a modular form f is lacunary if and only if f is a finite linear combination of CM forms.

The second type of special form we wish to consider are *newforms*. For our purposes we note they are eigenforms of all the Hecke operators and provide a basis for all cusp forms. By the theory developed by Serre and Swinnerton-Dyer [17] it is easy to prove certain types of congruences between their Fourier expansions and the expansions of Eisenstein series whose coefficients are given by special divisor functions. A famous example of such a congruence is

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

where $\sigma_{11}(n) = \sum_{d|n} d^{11}$.
Now we proceed by presenting new recurrences which follow from the theory discussed above.

If we consider the functions $\theta_O(z) = \sum_{n=1,odd}^{\infty} q^{n^2}$ and $\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$, we see that $\theta_O^d(z)$ has coefficient on q^n equal to the number of representations of n as a sum of d positive odd integers squared, $r_{d,odd}(n)$ and $\theta^d(z)$ has coefficient on q^n equal to the number of ways to write n as a sum of squares, $r_d(n)$. We now get a recurrence relation for these representation numbers that holds up to any fixed modulus for n on a set of density one in the integers.

Theorem 8. *Let m be a positive integer and let d be a positive odd integer. For $r_{d,odd}(n)$ the number of ways of representing n as a sum of d positive odd integers squared, and $r_d(n)$ the number of ways of representing n as a sum of d squares, we obtain*

$$r_{d,odd}(n-1) + \sum_{k=1}^{\infty} (-1)^k (r_{d,odd}(n-(6k+1)^2) + r_{d,odd}(n-(-6k+1)^2)) \equiv 0 \mod m$$

and

$$r_d(n-1) + \sum_{k=1}^{\infty} (-1)^k (r_d(n-(6k+1)^2) + r_d(n-(-6k+1)^2)) \equiv 0 \mod m$$

for almost all n (i.e. for a set of n with density 1 in the set of integers).

Proof. As a quotient of η-functions, we may rewrite (8) as

$$\theta_O(z) = \frac{\eta^2(16z)}{\eta(8z)}.$$

Therefore the generating function for $r_{d,odd}(n)$ is given by

$$\theta_O^d(z) = \frac{\eta^{2d}(16z)}{\eta^d(8z)},$$

a modular form of weight $\frac{d}{2}$. By Euler's Pentagonal Number Formula it turns out that

$$\eta(24z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2}.$$

Therefore multiplying $\theta_O^d(z)$ by $\eta(24z)$ yields

$$\theta_O^d(z)\eta(24z) = \frac{\eta^{2d}(16z)\eta(24z)}{\eta^d(8z)} = \sum_{n=1}^{\infty} a(n)q^n,$$

a cusp form of integer weight $\frac{d+1}{2}$. By Serre's divisibility theorem we observe that $a(n) \equiv 0 \mod m$ for almost all n given any fixed positive integer m. In terms of $r_{d,odd}(n)$, we find that

$$\sum_{n=1}^{\infty} r_{d,odd}(n)q^n \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2} = \sum_{n=1}^{\infty} a(n)q^n.$$

The proof for $r_{d,odd}(n)$ now follows. An analogous proof works for $r_d(n)$. □

Similarly we get a recurrence relation for $r_{d,odd}(n)$ and for $r_d(n)$ by multiplying by the Jacobi Triple Product rather than the Euler Pentagonal Number Formula.

When one can multiply a generating function by $\eta(az)$ or $\eta^3(az)$ and obtain a lacunary power series, we obtain recurrences which hold for almost all n. An example of this phenomenon is given in the following theorem.

Theorem 9. *Let $r_{3,odd}(n)$ be the number of ways of representing n as a sum of three odd squares. Then*

$$r_{3,odd}(n-1) + \sum_{k=1}^{\infty} (-1)^k (2k+1) r_{3,odd}(n-(2k+1)^2) = 0$$

for almost all n.

Proof. As in Theorem 8, we interpret the generating function for $r_{3,odd}(n)$ as an η-quotient. In particular we note that

$$\theta_O^3(z) = \frac{\eta^6(16z)}{\eta^3(8z)}.$$

By (2) we note that

$$\eta^3(8z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2}.$$

The formal product

$$\theta_O^3(z)\eta^3(8z) = \eta^6(16z) = \sum_{n=1}^{\infty} a(n)q^n$$

is a weight 3 cusp form; moreover by Serre's Theorem with $r = 3$ it turns out that $a(n) = 0$ for almost all n. Therefore we find that

$$\sum_{n=1}^{\infty} r_{3,odd}(n)q^n \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2} = \sum_{n=1}^{\infty} a(n)q^n,$$

where $a(n) = 0$ for almost all n. The result now follows easily by the formal product of two power series. $\square$

We now consider t-core partitions. Recall, that the generating function for the number of t-core partitions is

$$\sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{tn})^t}{1-x^n}.$$

We get

Theorem 10. *For $c_t(n)$ the number of t-core partitions of n,*

$$c_t(n) + \sum_{k=1}^{\infty} (-1)^k (c_t(n-\omega(k)) + c_t(n-\omega(-k))) = 0$$

for all n such that $n \not\equiv 0 \mod t$.

Proof. Since

$$\prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n} \prod_{n=1}^{\infty} 1-q^n = \prod_{n=1}^{\infty} (1-q^{tn})^t$$

we see the only powers of q on the right are powers of q^t, but on the left we have

$$\left(\sum_{n=0}^{\infty} c_t(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}}\right).$$

Comparing coefficients on like powers of q we get the result. $\square$

For certain t we do much better by Serre's classification of CM η-products.

Theorem 11. *For $t = 2, 4, 6, 8, 10, 14,$ or 26 we get*

$$c_t(n) + \sum_{k=1}^{\infty} (-1)^k (c_t(n-\omega(k)) + c_t(n-\omega(-k))) = 0$$

for almost all n.

Note that this tells us that even for $n \equiv 0 \mod t$ we almost always get this recurrence.

Proof. Multiplying the generating function for $c_t(n)$ by Euler's Pentagonal Number Formula we obtain

$$\left(\prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}\right)\left(\prod_{n=1}^{\infty} 1-q^n\right) = \prod_{n=1}^{\infty} (1-q^{tn})^t.$$

Since the right hand side is, up to change of variable, the sort of product in Serre's classification, this means almost all coefficients are in fact zero. Comparing coefficients on q^n we now get the result. $\square$

In some cases we can be very explicit and improve even on our last theorem

Theorem 12. *For $c_3(n)$ the number of 3-core partitions of n*

$$c_3(n)+\sum_{k=1}^{\infty}(-1)^k(c_3(n-\omega(k))+c_3(n-\omega(-k)))=\begin{cases}0 \text{ if } n\neq \frac{3m^2+3m}{2}\\ (-1)^m(2m+1) \text{ if } n=\frac{3m^2+3m}{2}\end{cases}.$$

Proof. Since

$$\left(\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3}{1-q^n}\right)\left(\prod_{n=1}^{\infty}1-q^n\right)=\prod_{n=1}^{\infty}(1-q^{3n})^3$$

we can realize the generating function for the t-core in terms of Euler's Pentagonal Number Formula and Jacobi's Triple Product; namely

$$\left(\sum_{n=0}^{\infty}c_3(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}\right)=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{3n^2+3n}{2}}.$$

Comparing coefficients of like powers of q we obtain the result. □

Finally we get some recurrences that involve the number of points on elliptic curves over finite fields, divisor functions, and the Ramanujan τ-function. Recall that an elliptic curve E is given by $y^2=x^3+ax+b$. The discriminant of E, denoted Δ, is defined by $\Delta := -4a^3-27b^2$. If $p\nmid\Delta$ is a prime, then the elliptic curve E has *good reduction* mod p. For such primes we let N_p denote the number of points on the elliptic curve E over F_p, the finite field with p elements. In particular N_p is one more than the number of solutions to the congruence

$$y^2\equiv x^3+ax+b \mod p.$$

The extra point corresponds to the additional point at infinity which does not correspond to an affine point (x,y).

The Hasse-Weil $L-$function $L(E,s)$ is central to the analytic and algebraic theory of elliptic curves. This function is constructed as an Euler product over primes like the Riemann $\zeta-$function. Given N_p, we define $a(p)$ by $a(p):=p+1-N_p$. For those primes where E has good reduction, we define the p^{th} factor of the Euler product of $L(E,s)$ by

$$\frac{1}{1-a(p)p^{-s}+p^{1-2s}}.$$

Precisely we obtain

$$L(E,s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\prod_{p\mid\Delta}\frac{1}{1-a(p)p^{-s}}\prod_{p\nmid\Delta}\frac{1}{1-a(p)p^{-s}+p^{1-2s}}.$$

For elliptic curves over $\mathbb{Q}$ (i.e. $a,b\in\mathbb{Q}$), the Taniyama-Weil Conjecture asserts that the Mellin transform of $L(E,s)$ is a newform of weight 2. In other words the function $f(z)=\sum_{n=1}^{\infty}a(n)q^n$ is a newform of weight 2 with respect to some congruence subgroup $\Gamma_0(N)$. For elliptic curves with complex multiplication Deuring and Shimura [3,16] proved this conjecture; moreover they proved that the Mellin transform of $L(E, s)$ is a modular form with complex multiplication. For more on elliptic curves see [7].

In the next two theorems we obtain recurrences which relate the number of $2-$core and $4-$core partitions with the $L-$function of two basic elliptic curves with complex multiplication. Although the next theorem is a recurrence involving the number of $2-$core partitions of n, we note that $c_2(n) = 0$ if n is not a triangular number and $c_2(n) = 1$ if n is a triangular number. This follows from Jacobi's formula

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n=0}^{\infty} c_2(n)q^n = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}.$$

Theorem 13. *For $c_2(n)$ the number of 2-core partitions of n and $a(n)$ the coefficients for the L-function of $y^2 = x^3 - x$ we get*

$$c_2(n) + \sum_{k=1}^{\infty}(-1)^k(2k+1)c_2(n - \frac{k(k+1)}{2}) = a(4n+1).$$

Moreover we obtain

$$c_2(n) + \sum_{k=1}^{\infty}(-1)^k(2k+1)c_2(n - \frac{k(k+1)}{2}) = 0$$

for almost all n.

Proof. The elliptic curve $y^2 = x^3 - x$ has L-function corresponding to the weight 2 cusp form

$$\eta^2(4z)n^2(8z) = q\prod_{n=1}^{\infty}(1-q^{8n})^2(1-q^{4n})^2.$$

For details, see [2]. But the 2-core generating function times the Jacobi Triple Product Identity is

$$\left(\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{1-q^n}\right)\left(\prod_{n=1}^{\infty}(1-q^n)^3\right) = \left(\sum_{n=0}^{\infty}c_2(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}\right).$$

Making the change of variables $q \to q^4$ and multiplying by q we get the L-function of the elliptic curve on the left hand side, so

$$\sum_{n=1}^{\infty}a(n)q^n = q\left(\sum_{n=0}^{\infty}c_2(n)q^{4n}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{2n^2+2n}\right).$$

Comparing like terms we get the first result; the second result follows since the modular form $\eta^2(4z)\eta^2(8z)$ is a CM form so almost all of its coefficients vanish.

□

Here is the other theorem where a recurrence relation is established between $t-$core partitions and elliptic curves.

Theorem 14. *For $c_4(n)$ the number of 4-core partitions of n and $a(n)$ the coefficients in the L-function of $y^2 = x^3 + 1$,*

$$c_4(n) + \sum_{k=1}^{\infty}(-1)^k(c_4(n - \frac{3k^2+k}{2}) + c_4(n - \frac{3k^2-k}{2})) = a(\frac{3n}{2} + 1).$$

Moreover we obtain

$$c_4(n) = \sum_{k=1}^{\infty}(-1)^k(c_4(n - \frac{3k^2+k}{2}) + c_4(n - \frac{3k^2-k}{2})) = 0$$

for almost all n.

Proof. It is well known [2] that the weight 2 cusp form corresponding to the L-function of the elliptic curve $y^2 = x^3 + 1$ is $\eta^4(6z) = q\prod_{n=1}^{\infty}(1 - q^{6n})^4$. The η-quotient $\frac{\eta^4(96z)}{\eta(24z)}$ in terms of $c_4(n)$ is

$$\frac{\eta^4(96z)}{\eta(24z)} = \sum_{n=0}^{\infty} c_4(n)q^{24n+15}.$$

Since $\eta(24z)$ has the Fourier expansion

$$\eta(24z) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2},$$

we find that

$$\sum_{n=0}^{\infty} c_4(n)q^{24n+15} \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2} = \eta^4(96z).$$

So if $\eta^4(6z) = \sum_{n=1}^{\infty} a(n)q^n$, then

$$\sum_{n=0}^{\infty} c_4(n)q^{24n+15} \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2} = \sum_{n=1}^{\infty} a(n)q^{16n}.$$

Therefore we obtain

$$\sum_{n=0}^{\infty} c_4(n)q^{24n} \sum_{n=-\infty}^{\infty}(-1)^n q^{(36n^2+12n)} = \sum_{n=1}^{\infty} a(n)q^{16n-16}.$$

The first result now follows by the formal product of power series. The second result follows from Serre's theorem where $r = 2$. $\square$

Now we get congruential recurrence relations between divisor functions and certain t-cores. There are several well known congruences for the Ramanujan τ- function. But the τ-function is by definition

$$\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{n=1}^{\infty}(1 - q^n)^{24}.$$

We can then relate the 24-core to τ.

Theorem 15. *Let $c_{24}(n)$ be the number of 24-core partitions of n and $\tau(n)$ the Ramanujan τ-function. Then*

$$c_{24}(n)+\sum_{k=1}^{\infty}(-1)^k(c_{24}(n-\omega(k))+c_{24}(n-\omega(-k)))=\begin{cases}\tau(1+\frac{n}{24}) \text{ if } n\equiv 0 \mod 24\\ 0 \text{ otherwise.}\end{cases}$$

Proof. Since, by definition,

$$\sum_{n=1}^{\infty}\tau(n)q^n=q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

and

$$\left(\sum_{n=0}^{\infty}c_{24}(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{3n^2+n}{2}}\right)=\prod_{n=1}^{\infty}\frac{(1-q^{24n})^{24}}{1-q^n}(1-q^n)$$

we get

$$\left(\sum_{n=0}^{\infty}c_{24}(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{3n^2+n}{2}}\right)=\sum_{n=1}^{\infty}\tau(n)q^{24n-24}.$$

Comparing like terms we get the result. □

Now we are interested in developing congruential recurrence relations for the number of 24−core partitions and the Ramanujan τ−function. Here we list several of the congruence properties of $\tau(n)$ [17]:

$$\tau(n)\equiv\sigma_{11}(n)\mod 691$$

$$\tau(n)\equiv n^{-30}\sigma_{71}(n)\mod 125 \text{ for } (n,5)=1$$

and

$$\tau(n)\equiv n\sigma_9(n)\mod 7$$

which now tell us something about the 24-core.

Corollary 1. *For $c_{24}(n)$ the number of 24-core partitions of n and $\sigma_d(n)$ the divisor function*

$$c_{24}(24n)+\sum_{k=1}^{\infty}(-1)^k(c_{24}(24n-\omega(k))+c_{24}(24n-\omega(-k)))\equiv\sigma_{11}(1+n)\mod 691$$

$$c_{24}(24n)+\sum_{k=1}^{\infty}(-1)^k(c_{24}(24n-\omega(k))+c_{24}(24n-\omega(-k)))\equiv(1+n)^{-30}\sigma_{71}(1+n)\mod 5^3$$

for $n\not\equiv -1 \mod 5$ and

$$c_{24}(24n)+\sum_{k=1}^{\infty}(-1)^k(c_{24}(24n-\omega(k))+c_{24}(24n-\omega(-k)))\equiv(1+n)\sigma_9(1+n)\mod 7.$$

Proof. These (and others) follow from the last theorem and congruences for $\tau(n)$. □

As a final example, since $c_{12}(n)$ is related to $\eta^{12}(2z)$, we can use this recurrence and known congruences of $\eta^{12}(2z)$ to get

Theorem 16. *For $c_{12}(n)$ the number of 12-core partititions of n,*

$$\left(\sum_{n=0}^{\infty} c_{12}(n)q^{n+6}\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}\right) = q^6\prod_{n=1}^{\infty}\frac{(1-q^{12n})^{12}}{1-q^n}\prod_{n=1}^{\infty}1-q^n = \eta^{12}(12z).$$

So under a simple change of variables we can transfer congruences about the coefficients of $\eta^{12}(2z)$ to those for $c_{12}(n)$. For example

$$c_{12}(12n)+\sum_{k=1}^{\infty}(-1)^k(c_{12}(12n-\omega(k))+c_{12}(12n-\omega(-k)))\equiv\sigma_5(2n+1) \mod 256.$$

Proof. The first formula follows from the definition. Replacing q by $q^{\frac{1}{6}}$ and equating coefficients of like integral powers of q we get $c_{12}(6n)+\sum_{k=1}^{\infty}(-1)^k(c_{12}(6n-\omega(k)) + c_{12}(6n-\omega(-k))) = a(1+n)$ for $a(1+n)$ the coefficient on q^{1+n} in $\eta^{12}(2z)$. The recurrence then follows from $a(2n+1)\equiv\sigma_5(2n+1) \mod 256$ [10].

□

4. Acknowledgements

The authors would like to thank George Andrews for his help in the preparation of this paper.

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