DIVISIBILITY OF CERTAIN PARTITION FUNCTIONS BY POWERS OF PRIMES

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Dedicated to the memory of Nathan Fine

Abstract. Let $k = {p_1}^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of a positive integer k and let $b_k(n)$ denote the number of partitions of a non-negative integer n into parts none of which are multiples of k. If M is a positive integer, let $S_k(N;M)$ be the number of positive integers $n \leq N$ for which $b_k(n) \equiv 0 \pmod{M}$. If $p_i^{a_i} \geq \sqrt{k}$, we prove that, for every positive integer j

$$\lim_{N\to\infty} \frac{S_k(N;p_i^j)}{N} = 1.$$

In other words for every positive integer j, $b_k(n)$ is a multiple of p_i^j for almost every non-negative integer n. In the special case when $k = p$ is prime, then in representation-theoretic terms this means that the number of p-modular irreducible representations of almost every symmetric group S_n is a multiple of p^j. We also examine the behavior of $b_k(n) \pmod{p_i^j}$ where the non-negative integers n belong to an arithmetic progression. Although almost every non-negative integer $n \equiv r \pmod{t}$ satisfies $b_k(n) \equiv 0 \pmod{p_i^j}$, we show that there are infinitely many non-negative integers $n \equiv r \pmod{t}$ for which $b_k(n) \not\equiv 0 \pmod{p_i^j}$ provided that there is at least one such n. Moreover the smallest such n (if there are any) is less than $2 \cdot 10^8 p_i^{a_i+j-1} k^2 t^4$.

1. The main theorem

A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. The number of such partitions is denoted by $p(n)$, and the number of partitions where the summands are distinct is denoted by $q(n)$. If k is a positive integer, let $b_k(n)$ be the number of partitions of n into parts none which are multiples of k. It is known that $b_2(n) = q(n)$, and if p is prime, then $b_p(n)$ is the number of irreducible $p-$modular representations of the symmetric group S_n [7].

The generating function $G_k(x)$ for $b_k(n)$ is given by the infinite product:

$$G_k(x) := \sum_{n=0}^{\infty} b_k(n) x^n = \prod_{n=1}^{\infty} \frac{(1-x^{kn})}{(1-x^n)}. \tag{1}$$

In this paper we obtain some divisibility properties of $b_k(n)$ by powers of certain special primes using the theory of modular forms as developed by Serre [15]. For more on modular forms see [8].

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If M is a positive integer, then let $S_k(N;M)$ be the number of positive integers $n \leq N$ for which $b_k(n) \equiv 0 \pmod{M}$. Thus for a positive integer N, the ratio

$$\frac{S_k(N;M)}{N}$$

is the arithmetic density of the set of positive integers $n \leq N$ for which $b_k(n) \equiv 0 \pmod{M}$.

Recently Alladi [1] has obtained combinatorial proofs of the fact that $\lim_{N\to\infty} \frac{S_2(N;2^k)}{N} = 1$ for small values of k. His methods involve the new theory of partition identities involving weights and gaps. Throughout this paper (unless otherwise stated), all power series $\sum_{n=0}^{\infty} a(n)x^n$ are assumed to have integer coefficients, and

$$\sum_{n=0}^{\infty} a(n)x^n \equiv \sum_{n=0}^{\infty} b(n)x^n \pmod{M}$$

means that $a(n) \equiv b(n) \pmod{M}$ for all n.

First we show that $\lim_{N\to\infty} \frac{S_2(N;2)}{N} = 1$ follows from a classical result in the theory of partitions. Since $(1 - X^n) \equiv (1 + X^n) \pmod{2}$, we find by Euler's Pentagonal Number Theorem that

$$G_2(x) = \sum_{n=0}^{\infty} b_2(n)x^n = \prod_{n=1}^{\infty}(1+x^n) \equiv \prod_{n=1}^{\infty}(1-x^n) \equiv \sum_{n\in\mathbb{Z}} x^{\frac{3n^2+n}{2}} \pmod{2}.$$

Hence it is clear that the set of non-negative integers n for which $b_2(n)$ is odd has density zero.

More generally in [6], it was proved that for every prime p that the set of non-negative integers n for which $b_p(n) \equiv 0 \pmod{p}$ has density 1. We show that this phenomenon also holds in many other cases. We prove:

Theorem 1. *Let $k = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of a positive integer k and let $b_k(n)$ denote the number of partitions of n into parts none of which are multiples of k. If $p_i^{a_i} \geq \sqrt{k}$, then for every positive integer j*

$$\lim_{N\to\infty} \frac{S_k(N;p_i^j)}{N} = 1.$$

In other words the set of those positive integers n for which $b_k(n) \equiv 0 \pmod{p_i^j}$ has arithmetic density one. In fact there exists a positive constant α depending on $p_i, j,$ and k such that there are at most $O\left(\frac{N}{log^{\alpha} N}\right)$ many integers $n \leq N$ for which $b_k(n)$ is not divisible by p_i^j.

Proof. We first note that if $f(x) = 1 + \sum_{n=1}^{\infty} a(n)x^n$ is a power series with integer coefficients such that $a(n) \equiv 0 \pmod{p}$ for all $n \geq 1$, then $f^{p^j}(x)$, the p^j power of $f(x)$, satisfies

$$f^{p^j}(x) \equiv 1 \pmod{p^{j+1}}.$$

By hypothesis this holds for $j = 0$, and if $f^{p^j}(x) = 1 + p^{j+1}g(x)$, then

$$f^{p^{j+1}}(x) = [1 + p^{j+1}g(x)]^p = 1 + p^{j+2}h(x),$$

completing the induction.

Now recall that the Dedekind eta function $\eta(\tau)$, is defined by

$$\eta(\tau) := x^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-x^n) \quad \text{(here } x := e^{2\pi i\tau}\text{throughout).}$$

We note that $\eta(24\tau) = x\prod_{n=1}^{\infty}(1-x^{24n})$ is a power series in x. Define $f_i(\tau)$

$$f_i(\tau) = \prod_{n=1}^{\infty} \frac{(1-x^{24n})^{p_i^{a_i}}}{(1-x^{24p_i^{a_i}n})} = \frac{\eta^{p_i^{a_i}}(24\tau)}{\eta(24p_i^{a_i}\tau)}.$$

Since $(1-X)^{p_i^{a_i}} \equiv (1-X^{p_i^{a_i}}) \pmod{p_i}$, we find that $f_i(\tau) \equiv 1 \pmod{p_i}$. Therefore by the remarks above we have

$$f_i^{p_i^j}(\tau) = \frac{\eta^{p_i^{a_i+j}}(24\tau)}{\eta^{p_i^j}(24p^{a_i}\tau)} \equiv 1 \pmod{p_i^{j+1}}. \tag{2}$$

Define $F_{i,j,k}(\tau)$ by

$$F_{i,j,k}(\tau) := \frac{\eta(24k\tau)}{\eta(24\tau)} \cdot \left(\frac{\eta^{p_i^{a_i}}(24\tau)}{\eta(24p_i^{a_i}\tau)}\right)^{p_i^j}.$$

Modulo p_i^{j+1}, we have

$$F_{i,j,k}(\tau) = \frac{\eta(24k\tau)}{\eta(24\tau)} f_i^{p_i^j}(\tau) \equiv \frac{\eta(24k\tau)}{\eta(24\tau)} = x^{k-1}\prod_{n=1}^{\infty}\frac{(1-x^{24kn})}{(1-x^{24n})} \pmod{p_i^{j+1}}.$$

Therefore by (1) we have

$$F_{i,j,k}(\tau) \equiv \sum_{n=0}^{\infty} b_k(n)x^{24n+k-1} \pmod{p_i^{j+1}}. \tag{3}$$

Now we briefly recall the modular properties of products of Dedekind eta functions. Let $f(\tau)$ be such a product defined by

$$f(\tau) = \prod_{0<\delta|N} \eta^{r(\delta)}(\delta\tau)$$

where N is a positive integer and $r(\delta) \in \mathbb{Z}$. If

$$\sum_{\delta|N} \delta r(\delta) \equiv 0 \pmod{24},$$

$$\sum_{\delta|N} \frac{Nr(\delta)}{\delta} \equiv 0 \pmod{24},$$

and $w := \frac{1}{2}\sum_{\delta|N} r(\delta)$ is a positive integer, then $f(\tau)$ is a modular form of weight w with Nebentypus character $\chi(d) = \left(\frac{(-1)^w D}{d}\right)$ with respect to the group $\Gamma_0(N)$ where $D := \prod_{\delta|N}\delta^{r(\delta)}$. Moreover since $\eta(\tau)$ does not vanish on the upper half of the complex plane, $f(\tau)$ is a holomorphic modular form if it is holomorphic at all of the cusps of $\Gamma_0(N)$.

To determine whether an eta-product is holomorphic at the cusps of $\Gamma_0(N)$ we use formulas that were shown in [4,9]. The order of the eta-product $f(\tau)$ at the cusp $\frac{c}{d}$ where c and d are positive integers and $d \mid N$ is

$$\frac{N}{24d\left(d, \frac{N}{d}\right)} \sum_{\delta \mid N} \frac{(d, \delta)^2 r(\delta)}{\delta}. \tag{4}$$

Since every cusp of $\Gamma_0(N)$ can be represented by such a fraction $\frac{c}{d}$, one has little trouble determining whether an eta-product is a holomorphic modular form.

From the above discussion, it is easily seen that $F_{i,j,k}(\tau)$ is a form of weight $w = \dfrac{p_i^j(p_i^{a_i}-1)}{2}$ on $\Gamma_0(576k)$ with $r(24k) = 1, r(24) = p_i^{a_i+j} - 1, r(24p_i^{a_i}) = -p_i^j$, and $r(\delta) = 0$ for all other $\delta \mid 576k$. Note that if p_i is an odd prime, or if $p_i = 2$ and $j \geq 1$, then w is a positive integer.

By (4) it follows that $F_{i,j,k}(\tau)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{(d, 24k)^2}{24k} + \frac{\left(p_i^{a_i+j}-1\right)(d, 24)^2}{24} - \frac{p_i^j (d, 24p_i^{a_i})^2}{24p_i^{a_i}} \geq 0.$$

After clearing denominators this inequality becomes

$$p_i^{a_i}(d, 24k)^2 + (kp_i^{2a_i+j} - kp_i^{a_i})(d, 24)^2 - kp_i^j(d, 24p_i^{a_i})^2 \geq 0,$$

which can be rewritten as

$$\frac{(kp_i^{2a_i+j} - kp_i^{a_i})(d, 24)^2}{(d, 24p_i^{a_i})^2} + \frac{p_i^{a_i}(d, 24k)^2}{(d, 24p_i^{a_i})^2} \geq kp_i^j.$$

Since $\dfrac{(d, 24)^2}{(d, 24p_i^{a_i})^2} \geq \dfrac{1}{p_i^{2a_i}}$ and $\dfrac{(d, 24k)^2}{(d, 24p_i^{a_i})^2} \geq 1$, we find that $F_{i,j,k}(\tau)$ is holomorphic at the cusp $\frac{c}{d}$ if

$$kp_i^j - \frac{k}{p_i^{a_i}} + p_i^{a_i} \geq kp_i^j,$$

that is, if $p_i^{a_i} \geq \frac{k}{p_i^{a_i}}$, or finally $p_i^{a_i} \geq \sqrt{k}$. Since this holds by hypothesis, $F_{i,j,k}(\tau)$ is holomorphic on $\Gamma_0(576k)$.

Without loss of generality we can assume that $j \geq 1$, and hence that $F_{i,j,k}(\tau)$ is a holomorphic integer weight form. By a theorem of Serre [15], if $F(\tau) = \sum_{n=0}^{\infty} b(n)x^n$ is a holomorphic form of positive integer weight on a congruence subgroup of $SL_2(\mathbb{Z})$ where the $b(n)$ are integers, then there is a positive constant α such that there are at most $O(\frac{N}{log^{\alpha} N})$ many integers $n \leq N$ for which $b(n)$ is not divisible by M. In particular almost every coefficient $b(n)$ is a multiple of M.

Theorem 1 follows from this result with $M = p_i^{j+1}$, since by (3)

$$F_{i,j,k}(\tau) \equiv \sum_{n=0}^{\infty} b_k(n)x^{24n+k-1} \pmod{p_i^{j+1}}.$$

$\square$

As noted above, when $k = p$ is prime, $b_p(n)$ is the number of $p-$modular irreducible representations of the symmetric group S_n. Therefore we obtain the following corollary:

Corollary 1. *If p is prime and j is a positive integer, then the number of $p-$modular irreducible representations of almost every symmetric group S_n is a multiple of p^j.*

2. A finer investigation

Although for any positive integer j the set of non-negative integers n for which $b_k(n) \equiv 0 \pmod{p_i^j}$ has density one when $p_i^{a_i} \geq \sqrt{k}$, we now show that the set of those m for which $b_k(m) \not\equiv 0 \pmod{p_i^j}$ is indeed infinite. This will follow as a consequence of the fact that a holomorphic modular form cannot be congruent modulo M to a polynomial with constant term 0. We need a few preliminaries. If $f(x) = \sum_{n\geq N_0} a(n)x^n$ is a power series with integer coefficients and M is a positive integer, then we define $\mathrm{Ord}_M(f(x))$ to be the smallest n for which $a(n) \not\equiv 0 \pmod{M}$. If no such n exists, then we let $\mathrm{Ord}_M(f(x)) = +\infty$.

In [16] Sturm proved that if $f(\tau)$ and $g(\tau)$ are two holomorphic integer weight k forms with integer coefficients on some congruence subgroup Γ of $SL_2(\mathbb{Z})$, then $f(\tau) \equiv g(\tau) \pmod{p}$, where p is prime, if and only if

$$\mathrm{Ord}_p(f(\tau) - g(\tau)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma].$$

From this one can easily deduce the same criterion where the prime p is replaced by an arbitrary integer M. Now we may prove:

Theorem 2. *If $f(\tau) = \sum_{n=1}^{\infty} a(n)x^n \in M_k(N)$ for a pair of positive integers k and N with integer Fourier coefficients, then $f(\tau)$ is not congruent to a nontrivial polynomial modulo a positive integer M.*

Proof. Suppose that M is a positive integer for which

$$f(\tau) \equiv \sum_{n=1}^{T} a(n)x^n \pmod{M}.$$

Then if $p \equiv 1 \pmod{N}$ is prime, then

$$f(\tau) \mid T_p \equiv \sum_{n\geq 1} a(pn)x^n + p^{k-1}\sum_{n=1}^{T} a(n)x^{pn} \pmod{M}.$$

Define the constant C by

$$C := \max\left(\frac{k}{12}N^2\prod_{p\mid N}\left(1-\frac{1}{p^2}\right), T\right).$$

If $p \equiv 1 \pmod{N}$ is a prime where $p > C$ and $\gcd(p, M) = 1$, then $C < \mathrm{Ord}_M(f(\tau)|T_p) < +\infty$. By Sturm's theorem this implies that

$$f(\tau) \mid T_p \equiv 0 \pmod{M}.$$

However this is a contradiction because it is clear that

$$f(\tau) \mid T_p \equiv p^{k-1}f(p\tau) \not\equiv 0 \pmod{M}.$$

□

We now prove two corollaries.

Corollary 2. *Let $k = p_1^{a_1} p_2^{a_2} \cdot \cdot p_m^{a_m}$ be the prime factorization of a positive integer k, and let $b_k(n)$ denote the number of partitions of n into parts none of which are multiples of k. If $p_i^{a_i} \geq \sqrt{k}$, then there are infinitely many integers n for which*

$$b_k(n) \not\equiv 0 \pmod{p_i}.$$

Proof. It follows from (3) that the integer weight modular form $F_{i,1,k}(\tau)$ has Fourier expansion

$$F_{i,1,k}(\tau) \equiv \sum_{n=0}^{\infty} b_k(n) x^{24n+k-1} \equiv x^{k-1} + \dots \pmod{p_i^2}.$$

If there were only finitely many non-negative integers n for which $b_k(n) \not\equiv 0 \mod p_i$, then $F_{i,1,k}(\tau)$ would be congruent to a polynomial with constant term zero modulo p_i which contradicts Theorem 2. Hence there must be infinitely many such non-negative integers.

□

Although the set of non-negative integers for which $b_k(n) \not\equiv 0 \pmod{p_i}$ is quite thin when $p_i^{a_i} \geq \sqrt{k}$, there still are infinitely many such n. We now consider the case where the integers n lie in a given arithmetic progression. Although for almost every non-negative integer n in an arithmetic progression the integer $b_k(n)$ is divisible by any given power of p_i, we find that there are infinitely many non-negative integers in an arithmetic progression for which $b_k(n)$ is not a multiple of p_i^j if there is at least one such n. In other words, there is no constant $N(r, t, k, p_i, j)$ for which every $n > N(r, t, k, p_i, j)$ satisfies $b_k(tn + r) \equiv 0 \pmod{p_i^j}$, provided there is at least one $n \equiv r \pmod{t}$ for which the congruence does not hold. First note that there are examples of arithmetic progressions where congruences hold; for example the Ramanujan congruences for the ordinary partition function $p(n)$ modulo $5, 7$, and 11 imply that

$$b_5(5n + 4) \equiv 0 \pmod{5},$$

$$b_7(7n + 5) \equiv 0 \pmod{7},$$

and

$$b_{11}(11n + 6) \equiv 0 \pmod{11}$$

for every non-negative integer n.

Corollary 3. *Let $k = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of a positive integer k. If $p_i^{a_i} \geq \sqrt{k}$, then in any arithmetic progression $r \pmod{t}$, there are infinitely many integers $n \equiv r \pmod{t}$ for which $b_k(n) \not\equiv 0 \pmod{p_i^j}$ provided that there is at least one such n. Moreover provided there is one such n, then the smallest such n is less than $C(t, k, p_i)$ where*

$$C(t,k,p_i) = \begin{cases} \frac{2^{18} \cdot 3^6 p_i^{j-1} (p_i^{a_i} - 1) k^2 t^4}{d^2} \prod_{p \mid \frac{576 \cdot 24^2 k t^2}{d}} \left(1 - \frac{1}{p^2}\right) & \text{if } p_i \text{ is odd or } j \geq 2 \text{ if } p_i = 2 \\ \frac{2^{a_i+18} \cdot 3^6 k^2 t^4}{d^2} \prod_{p \mid \frac{576 \cdot 24^2 k t^2}{d}} \left(1 - \frac{1}{p^2}\right) & \text{if } p_i = 2 \text{ and } j = 1 \end{cases}$$

where $d = (24r + k - 1, 24t)$.

Proof. If $f(\tau) = \sum_{n=0}^{\infty} a(n) x^n$ is a holomorphic form of integer weight w on $\Gamma_0(N)$, then

$$f_{r,t}(\tau) := \sum_{n \equiv r \pmod{t}} a(n) x^n$$

is a holomorphic form of weight w on $\Gamma_1\left(\frac{Nt^2}{d}\right)$ where $d := \gcd(r,t)$. For a proof of this fact see [12]. The result essentially follows from the orthogonality relations of Dirichlet characters and the theory of twists of modular forms.

First assume that $p_i \neq 2$ or that $j \geq 2$ when $p_i = 2$. In these cases the form $F_{i,j-1,k}(\tau)$ is an integer weight $\frac{p_i^{j-1}(p_i^{a_i}-1)}{2}$ holomorphic modular form on $\Gamma_0(576k)$ satisfying

$$F_{i,j-1,k}(\tau) \equiv \sum_{n=0}^{\infty} b_k(n) x^{24n+k-1} \pmod{p_i^j}.$$

Now let $F_{i,j-1,k,r,t}(\tau)$ be the restriction of the Fourier expansion of $F_{i,j-1,k}(\tau)$ to those terms whose exponents are in the arithmetic progression $24r+k-1 \pmod{24t}$. Then $F_{i,j-1,k,r,t}(\tau)$ is a form of weight $\frac{p_i^{j-1}(p_i^{a_i}-1)}{2}$ on $\Gamma_1\left(\frac{576k\cdot 24^2t^2}{d}\right)$ where $d := (24r+k-1, 24t)$. Moreover $F_{i,j-1,k,r,t}(\tau)$ satisfies the congruence

$$F_{i,j-1,k,r,t}(\tau) \equiv \sum_{n\equiv r \pmod{t}} b_k(n) x^{24n+k-1} \pmod{p_i^j}.$$

Therefore by Sturm's theorem, if there is a non-negative integer $n \equiv r \pmod{t}$ for which $b_k(n) \not\equiv 0 \pmod{p_i^j}$, then the smallest such n satisfies

$$24n+k-1 \leq \frac{p_i^{j-1}(p_i^{a_i}-1)}{24}\left(\frac{576k\cdot 24^2t^2}{d}\right)^2 \prod_{p \,\mid\, \frac{576\cdot 24^2kt^2}{d}} \left(1-\frac{1}{p^2}\right)$$

where the product is over primes p. Now solving for n we find that if $b_k(n) \not\equiv 0 \mod p_i^j$, then

$$n \leq \frac{2^{18}\cdot 3^6 p_i^{j-1}(p_i^{a_i}-1)k^2t^4}{d^2} \prod_{p \,\mid\, \frac{576\cdot 24^2kt^2}{d}} \left(1-\frac{1}{p^2}\right) - \frac{k}{24} + \frac{1}{24}.$$

The remaining case to consider is where $p_1 = 2$ and $j = 1$. Here $F_{1,0,k}(\tau)$ is a weight $w = 2^{a_1-1} - \frac{1}{2}$ holomorphic modular form with respect to $\Gamma_0(576k)$. Since $\Theta(\tau) = 1 + 2\sum_{n=1}^{\infty} x^{n^2} \equiv 1 \pmod{2}$ is a weight $\frac{1}{2}$ modular form with respect to $\Gamma_0(4)$, if we replace $F_{1,0,k}(\tau)$ by $F_{1,0,k}(\tau)\Theta(\tau)$, we obtain a holomorphic integer weight 2^{a_1-1} form with respect to $\Gamma_0(576k)$.

Now repeating the above argument with this form we find that the smallest non-negative integer $n \equiv r \pmod{t}$ for which $b_k(n)$ is odd (if there are any), satisifes

$$n \leq \frac{2^{a_i+18}\cdot 3^6k^2t^4}{d^2} \prod_{p \,\mid\, \frac{576\cdot 24^2kt^2}{d}} \left(1-\frac{1}{p^2}\right) - \frac{k}{24} + \frac{1}{24}$$

where $d := (24r+k-1, 24t)$.

□

It is easy to verify that the bounds given in Corollary 3 imply the bound given in the abstract.

3. Final Remarks

There are many other partition functions for which similar methods yield interesting divisibility properties. One simply needs to search for a generating function which is congruent to the Fourier expansion of an integer weight holomorphic modular form on some congruence subgroup of $SL_2(\mathbb{Z})$; then one can apply Serre's Theorem. In this paper we constructed the modular forms $F_{i,j,k}(\tau)$.

To illustrate this general principle we briefly describe the situation in the case of the number of $t-$core partitions. If t is a positive integer, then a $t-$core partition of n is one where where the hook numbers of the associated Ferrers-Young graph are not multiples of t. Let $c_t(n)$ denote the number of such partitions (for more on $c_t(n)$ see [5,6,10,11]). Using the methods of this paper one can prove that if t is odd and m is a positive integer, then the set of non-negative integers n for which $c_t(n) \equiv 0 \pmod{m}$ has arithmetic density 1. This follows from the fact that the generating function for $c_t(n)$ is essentially a weight $\frac{t-1}{2}$ (which is an integer if t is odd) holomorphic modular form. Also there are infinitely many integers $n \equiv r \pmod{s}$ such that $c_t(n) \not\equiv 0 \pmod{M}$ provided that there is at least one such n. However the situation may be quite different when t is even.

One cannot expect to get density results for the ordinary partition function $p(n)$ by these methods. It is unlikely that there exists a positive integer $M > 1$ for which $p(n) \equiv 0 \pmod{M}$ for almost every non-negative integer n. It is of interest to note that simple questions regarding even the parity of $p(n)$ remain unresolved (see [11,12,13]). The obstruction to our understanding lies in the fact that the generating function for $p(n)$ is essentially $\eta^{-1}(24\tau)$, a weight $\frac{-1}{2}$ non-holomorphic modular form. Hence a better understanding of the Fourier coefficients of non-holomorphic integral and half-integral weight modular forms is required before significant progress can be made regarding the behavior of $p(n) \pmod{M}$. A preliminary investigation in this direction is contained in [3].

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