ODD VALUES OF THE PARTITION FUNCTION

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Abstract. Let $p(n)$ denote the number of partitions of an integer n. Recently the author has shown that in any arithmetic progression $r \pmod t$, there exist infinitely many N for which $p(N)$ is even, and there are infinitely many M for which $p(M)$ is odd, provided there is at least one such M. Here we construct finite sets of integers M_i for which $p(M_i)$ is odd for an odd number of i. Whereas Euler's recurrence allows us to find odd values of $p(n)$ when we already have one, the methods we describe do not rely on already having an odd value of $p(n)$.

A *partition* of a non-negative integer n is any non-increasing sequence of positive integers whose sum is n, and let $p(n)$ denote the number of partitions of n. Euler's generating function for $p(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{1}$$

He proved the following recurrence: for $n \geq 1$,

(2)

$$p(n)+\sum_{k=1}^{\infty}(-1)^k(p(n-\omega_1(k))+p(n-\omega_1(-k))) = 0. \quad (\text{where } \ \omega_1(k) := (3k^2+k)/2)$$

Thus if $p(n)$ is odd, then it is easy to find other odd values of the partition function.

Parkin and Shanks [22] conjectured that the number of $n \leq x$ for which $p(n)$ is odd is $\sim \frac{1}{2}x$. This is a very difficult conjecture, and it is not even known that there is a positive density of integers for which $p(n)$ is even (resp. odd). In 1966 Subbarao [24] made the following weaker conjecture:

Conjecture. *For any arithmetic progression $r \pmod t$, there are infinitely many integers $M \equiv r \pmod t$ for which $p(M)$ is odd, and there are infinitely many integers $N \equiv r \pmod t$ for which $p(N)$ is even.*

Using modular forms [19,20], the author has gone some way towards a proof of this conjecture. It is now known that in any arithmetic progression $r \pmod t$ there are infinitely many $N \equiv r \pmod t$ for which $p(N)$ is even, and there are infinitely

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many $M \equiv r \pmod{t}$ for which $p(M)$ is odd, provided there is at least one such M.

Unfortunately, the author has no modular form argument that guarantees the existence of a single odd value without making various assumptions. Hence it is of interest to develop means of constructing odd values of the partition function. In this note we show how this is easily done using known results concerning certain $t-$core partition functions. Similar arguments have already appeared in the literature (see [1,3,7,8,9,14,24]), and it is hoped that a complete understanding of such results will lead to the resolution of Subbarao's conjecture.

If t is a positive integer, then let $c_t(n)$ denote the number of $t-$core partitions of n. A partition is a *$t-$core* if none of the hook numbers of the associated Ferrers-Young diagram are multiples of t. Such partitions are rich with structure and are intimately connected to the representation theory of finite groups, and also to class groups of imaginary quadratic fields. For more on the arithmetic of $c_t(n)$ see [2,4,5,6,10,11,12, 13,14,15,17,18,21,23]).

The generating function for $c_t(n)$ is given by:

$$\sum_{n=0}^{\infty} c_t(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n} = \left(\sum_{n=0}^{\infty} p(n)q^n\right) \cdot \left(\sum_{n=0}^{\infty} a_t(n)q^n\right). \tag{3}$$

Here the coefficients $a_t(n)$ are given by the series $\sum_{n=0}^{\infty} a_t(n)q^n := \prod_{n=1}^{\infty}(1-q^{tn})^t$. Note that $a_t(n) = 0$ if t does not divide n.

If $t \geq 1$ is a positive integer, then for every positive integer n

$$c_t(n) = p(n) + \sum_{j=1}^{n} p(n-j)a_t(j). \tag{4}$$

Remark 1. If $t = 1$, then (4) is Euler's recurrence (2) since $c_1(n) = 0$ for every positive integer n, and $a_1(j) = (-1)^k$ if $j = \frac{3k^2+k}{2}$ for some integer k, and is 0 otherwise.

Now we give the following elementary results from which we obtain odd values of the partition function without having to start with an odd value. These results follow from (4) and known results concerning the parity of $c_t(n)$ when $t = 2, 3$, or 4. Similar results probably exist for certain $t \geq 5$, but it is not expected that they will be as elegant as those presented here.

Theorem 1. *If $n = \frac{m(m+1)}{2}$ and $\omega_2(k)$ is defined by*

$$\omega_2(k) := \begin{cases} \dfrac{k^2-1}{6} & \text{if } 1 \leq k \equiv 1, 5 \pmod{6}, \\ 0 & \text{otherwise}, \end{cases}$$

then an odd number of

$$p(n - \omega_2(1)),\ p(n - \omega_2(2)),\ p(n - \omega_2(3)) \ldots$$

are odd.

Proof. It is well known that $c_2(n) = 1$ if $n = \frac{m(m+1)}{2}$ for some m, and is 0 otherwise. Therefore by (4), if $n = \frac{m(m+1)}{2}$, then

$$1 = p(n) + \sum_{j=1}^{n} p(n-j)a_2(j). \tag{5}$$

Now,

$$\sum_{n=0}^{\infty} a_2(n)q^n = \prod_{n=1}^{\infty}(1-q^{2n})^2 \equiv \prod_{n=1}^{\infty}(1-q^{4n}) \equiv \sum_{1\le k\equiv 1,5 \pmod 6} q^{\frac{k^2-1}{6}} \pmod 2 \tag{6}$$

since

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{k=1}^{\infty}(-1)^k\left(q^{\frac{3k^2-k}{2}} + q^{\frac{3k^2+k}{2}}\right) \equiv \sum_{1\le k\equiv 1,5 \pmod 6} q^{\frac{k^2-1}{24}} \pmod 2,$$

and so $a_2(j)$ is odd if and only if $j = \frac{k^2-1}{6}$ where $1 \le k \equiv 1, 5 \pmod 6$. Hence by (5)

$$1 \equiv p(n) + \sum_{k=2}^{\infty} p(n-\omega_2(k)) \pmod 2,$$

and so an odd number of the following values are odd:

$$p(n) = p(n-\omega_2(1)),\ p(n-\omega_2(2)),\ p(n-\omega_2(3))\ldots.$$

□

Theorem 2. *If $n = 3m^2+2m$ or $3m^2-2m$ and $\omega_3(k)$ is defined by $\omega_3(k) := \frac{3k^2+3k}{2}$, then an odd number of*

$$p(n),\ p(n-\omega_3(1)),\ p(n-\omega_3(2)),\ p(n-\omega_3(3))\ldots$$

are odd.

Proof. By [6] it is known that

$$c_3(n) := \sum_{1\le d|3n+1} \left(\frac{d}{3}\right)$$

where $\left(\frac{d}{3}\right)$ is the usual Legendre symbol. Therefore $c_3(n)$ is odd if and only if $n = 3m^2+2m$ or $3m^2-2m$ for some non-negative integer m. By (4), if $n = 3m^2+2m$ or $3m^2-2m$, then

$$1 \equiv c_3(n) = p(n) + \sum_{k=1}^{\infty} p(n-k)a_3(k) \pmod 2.$$

Now, by a theorem of Jacobi,

$$\sum_{n=0}^{\infty} a_3(n)q^n = \prod_{n=1}^{\infty}(1-q^{3n})^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{3k^2+3k}{2}} \equiv \sum_{k=0}^{\infty} q^{\frac{3k^2+3k}{2}} \pmod 2, \tag{7}$$

and so $a_3(j)$ is odd if and only if $j = \frac{3k^2+3k}{2}$ for some non-negative integer k. The result follows.

□

Theorem 3. *If n is an integer for which*

$$8n+5 = p^{\alpha} N^2$$

where p is prime, $\alpha \equiv 1 \pmod 4$, and $gcd(p,N)=1$, and $\omega_4(k)$ is defined by

$$\omega_4(k) := \begin{cases} \dfrac{2k^2-2}{3} & \text{if } 1 \le k \equiv 1,5 \pmod 6 \\ 0 & \text{otherwise,} \end{cases}$$

then an odd number of

$$p(n-\omega_4(1)),\ p(n-\omega_4(2)),\ p(n-\omega_4(3))\ldots$$

are odd.

Proof. It immediately follows from [21, Theorem 3] that $c_4(n)$ is odd if $8n+5 = p^{\alpha}N^2$ where p is prime, $\alpha \equiv 1 \pmod 4$, and $\gcd(p,N)=1$. By (4), for such n we find that

$$1 \equiv c_4(n) = p(n) + \sum_{k=1}^{\infty} p(n-k)a_4(k) \pmod 2. \tag{8}$$

Now,

$$\sum_{n=0}^{\infty} a_4(n)q^n = \prod_{n=1}^{\infty}(1-q^{4n})^4 \equiv \prod_{n=1}^{\infty}(1-q^{16n}) \equiv \sum_{1\le k \equiv 1,5 \pmod 6} q^{\frac{2k^2-2}{3}} \pmod 2,$$

and so $a_4(j)$ is odd if and only if $j = \frac{2k^2-2}{3}$, where $1 \le k \equiv 1,5 \pmod 6$. The result follows.

□

Remark 2. To the author's knowledge this is the first time that the parity of $p(n)$ has been attached to prime factorizations of integers. Every integer n for which $8n+5$ is prime satisfies the conditions of the theorem, and the number of such $n \le x$ is $\sim \frac{2x}{\log 8x}$. However in the other theorems, the number of suitable $n \le x$ is $\sim c\sqrt{x}$, which is dominated by the above estimate. The conditions in Theorem 3 are somewhat special since the values of $c_4(n)$, and consequently parity, come from class numbers [21]. Combinatorially the parity of $c_4(n)$ can also be obtained by explicitly determining the number of self-conjugate $4-$cores of n. This is done in [10,21].

Remark 3. These three theorems may be used to construct larger and larger M for which $p(M)$ is odd. To construct odd $p(M)$ where $M > a$, let k and n be any pair of integers for which $a + \omega_t(k) < n < \omega_t(k+1)$. For suitable n, an odd number of the following values are odd:

$$p(n),\ p(n-\omega_t(1)),\ p(n-\omega_t(2)),\ldots .p(n-\omega_t(k))$$

By construction these odd values $p(M)$ satisfy $M > a$.

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