RAMANUJAN'S TERNARY QUADRATIC FORM

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Inventiones Mathematicae, 130, 3, 1997, pages 415-454.

1. Introduction

In [R], S. Ramanujan investigated the representation of integers by positive definite quadratic forms, and the ternary form

$$\phi_1(x,y,z) := x^2 + y^2 + 10z^2 \tag{1}$$

was of particular interest to him. This form is in a genus consisting of two classes, and

$$\phi_2(x,y,z) := 2x^2 + 2y^2 + 3z^2 - 2xz \tag{2}$$

is a representative for the other class. Ramanujan stated that [p.14,R]

"... *the even numbers which are not of the form* $x^2 + y^2 + 10z^2$ *are the numbers*

$$4^\lambda(16\mu + 6),$$

while the odd numbers that are not of that form, viz.,

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391 \ldots$$

do not seem to obey any simple law."

Following I. Kaplansky, we call a non-negative integer N *eligible* for a ternary form $f(x,y,z)$ if there are no congruence conditions prohibiting f from representing N. By the classical theory of quadratic forms, it is well known that any given genus of positive definite ternary quadratic forms represents every eligible integer. Consequently if a genus consists of a single class with representative $f(x,y,z)$, then f represents every eligible integer. In the case of Ramanujan's form, this only implies that an eligible integer, one not of the form $4^\lambda(16\mu + 6)$, is represented by ϕ_1 or ϕ_2.

1991 *Mathematics Subject Classification.* Primary 11E20; Secondary 11E25.

Key words and phrases. Ramanuajan's ternary quadratic form, modular forms, elliptic curves.
The first author is supported by NSF grants DMS-9508976 and DMS-9304580.

There are numerous examples of ternary forms f belonging to a genus with multiple classes where f represents every eligible integer. For instance the form $f(x,y,z) = x^2+3y^2+36z^2$ is in a genus with two classes and B. Jones and G. Pall [JP] have shown that f represents every eligible integer. Many other such examples may be found in [Di, Hs, Jo2, JP, Ka2, Ka3].

Motivated by these examples L. Dickson, B. Jones, G. Pall [Di, Jo1, Jo2, JP], among others initiated a study of ternary quadratic forms in an effort to explain and describe the behavior of ternary forms. From their work two categories of ternary forms emerged: *regular* ternary quadratic forms being those forms which represent all eligible integers, and *irregular* ternary quadratic forms being those forms which miss some eligible integers. In view of Ramanujan's list of exceptions ϕ_1 is irregular, whereas, as noted above, the form $f = x^2+3y^2+36z^2$ is regular.

There are various methods for determining whether a form is regular and if it turned out to be regular then the problem of the integers represented by it can be resolved. The situation is very different for irregular forms where there is no known effective way of determining the eligible integers which are represented, although individual arithmetic progressions (eg. $3n+2, 10n+5$) can be handled by an assortment of elementary methods. In fact the problem has never been resolved for any irregular form which misses at least two eligible integers. The powerful results of W. Duke and R. Schulze-Pillot [DS-P] dictate that if a large integer is represented by the spinor genus of a form then it is represented by the form itself. In our case this means that all but finitely many eligible integers are represented by both ϕ_1 and ϕ_2. As we shall see presently, this result depends on Siegel's lower bound for the class number of imaginary quadratic number fields, and is consequently ineffective. That is, it does not yield a bound beyond which there are no exceptions.

In this paper we address the question of finding all eligible numbers represented by Ramanujan's form ϕ_1. Besides the elements in Ramanujan's list there are the exceptions 679 and 2719 discovered by B. Jones, G. Pall, and H. Gupta [JP, Gu]. At our request W. Galway kindly verified that there are no further exceptions below $2 \cdot 10^{10}$. We are thus led to the following conjecture.

Conjecture. *The eligible integers which are not of the form $x^2+y^2+10z^2$ are:*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.$$

A complete resolution of this conjecture appears to be beyond the reach of present methods. One of the purposes of this paper is to provide a proof of the conjecture conditional on the truth of the Generalized Riemann Hypothesis (GRH).

It will be convenient below to consider only those N which are coprime to 10. Ramanujan noted that it suffices to consider the odd eligible integers, and to make the further reduction to integers coprime to 10 we need to eliminate the case $N \equiv 5 \pmod{10}$. Legendre proved that (see [p.261, Di2]) $2n+1 = x^2+y^2+2z^2$ has a solution for all non-negative integers n. Multiplying by 5 we see that

$$10n+5 = 5(x^2+y^2)+10z^2 = (2x+y)^2+(x-2y)^2+10z^2,$$

which verifies that every integer of the form $10n+5$ is represented by ϕ_1.

Therefore we may restrict our attention to those integers N coprime to 10. We now consider those eligible N which contain non-trivial square factors. If $N = \phi_1(x, y, z)$ is represented by ϕ_1, then $Nm^2 = \phi_1(xm, ym, zm)$ is represented for all $m \geq 1$. However if N is not represented by ϕ_1, then it is not clear whether any of the integers Nm^2 is represented. In this direction the results of J. Cassels, H. Davenport, and G.L. Watson [Ca, Da1, Da2, Wat] on indefinite quadratic forms imply that if N is an eligible integer, then Nm^2 is represented by ϕ_1 for some $m \leq (27N)^{3/2}$. More recently, by the work of J. Benham and J. Hsia [BHs] (which has been extended by J. Hsia and M. Jöchner [HsJ]), it is now known that all eligible integers not of the form $x^2 + y^2 + 10z^2$ are square-free. We obtain the same result using a completely different argument which suggests our strategy for dealing with square-free eligible integers.

Theorem 1. *If N is an eligible integer which is not square-free, then it is of the form $x^2 + y^2 + 10z^2$.*

We now know that every non-represented eligible integer N must be square-free and relatively prime to 10. Let $r_i(N)$ denote the number of representations of N by ϕ_i, and let $R_i(N)$ denote the number of *primitive* representations of N by ϕ_i. Recall that a representation $\phi_i(x, y, z) = N$ is called primitive if $\gcd(x, y, z) = 1$. We will see, in Proposition 1 below, that for any positive eligible integer N which is coprime to 10, $R_1(N)/2 + R_2(N) = h(-40N)$. In conjunction with Theorem 1 this observation immediately yields the following corollary which may be viewed as the "simple law" that Ramanujan sought.

Corollary 1. *Let N be an eligible integer. Then N is not of the form $x^2 + y^2 + 10z^2$ if and only if N is a square-free integer coprime to 10, and*

$$R_2(N) = h(-40N).$$

To describe our results for square-free N we need some notation. Let E be the elliptic curve

$$E: \qquad y^2 = x^3 + x^2 + 4x + 4, \tag{3}$$

and for every integer D we let $E(D)$ denote the $D-$quadratic twist of E; namely the curve given by:

$$E(D): \qquad y^2 = x^3 + Dx^2 + 4D^2x + 4D^3. \tag{4}$$

Theorem 2. *Let N be a square-free eligible integer. If N is not of the form $x^2+y^2+10z^2$, then*

$$h^2(-40N) = \frac{4\sqrt{N}}{\Omega(E(-10))} L(E(-10N), 1),$$

where $\Omega(E(-10)) \sim 0.71915$ is the real period of $E(-10)$.

Corollary 2. *Let N be a square-free eligible integer. Assuming the Conjectures of Birch and Swinnerton-Dyer, if N is not of the form $x^2+y^2+10z^2$, then*

$$h^2(-40N) = |\text{Ш}(E(-10N))| \cdot \prod_p \omega_p(E(-10N)).$$

Here Ш$(E(-10N))$ is the Tate-Shafarevich group, and the $\omega_p(E(-10N))$ are local Tamagawa factors.

Let $a(n) = (r_1(n) - r_2(n))/4$ and put $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Then $f(z)$ is a weight $3/2$ cusp form. If a square-free eligible integer N is not represented by Ramanujan's form then, by Corollary 1, $|a(N)| = r_2(N)/4 = R_2(N)/4 = h(-40N)/4$. By Siegel's (ineffective) lower bound for class numbers we obtain that $|a(N)| \gg N^{1/2-\epsilon}$. The trivial upper bound for Fourier coefficients of cusp forms gives $|a(N)| \ll N^{1/2+\epsilon}$, which barely fails to give useful information in our problem. Plainly if one could go a little below the trivial upper bound, that is obtain $|a(N)| \ll N^{1/2-\delta}$ for some fixed $\delta > 0$, it would follow (ineffectively) that there are only finitely many exceptions to Ramanujan's form. Observe that the Ramanujan conjectures for half-integral weight cusp forms (as yet unproved) predict that $|a(N)| \ll N^{1/4+\epsilon}$. Although the Ramanujan conjectures appear out of reach at the present, a remarkable breakthrough has been achieved by H. Iwaniec [I] and W. Duke [Du] who go below the trivial bound. Iwaniec's work obtains upper bounds for the Fourier coefficients of cusp forms of weight $\geq 5/2$ while a particular case of Duke's results (for spectral Maass forms) covers forms of weight $3/2$. These results yield $|a(N)| \ll \tau(N)N^{3/7}(\log 2N)^2$ where $\tau(N)$ ($\ll N^{\epsilon}$) is the number of divisors of N: consequently at most finitely many eligible integers are missed by Ramanujan's form. With current technology it does not appear possible to make this conclusion effective. The known effective lower bounds for class numbers, due to D. Goldfeld [Go] and B. Gross and D. Zagier [GZ], only imply that $|a(N)| \gg \log N$ which is too feeble for our purposes.

Assuming the Riemann hypothesis for Dirichlet's L-functions, J. E. Littlewood [L] has shown effectively that $h(-40N) \gg \sqrt{N}/\log\log N$. This result, which is best possible up to constants, provides a conditional, effective resolution to our problem. Unfortunately the bound for N, beyond which every eligible integer is represented, obtained in this fashion is enormous. To illustrate this let us assume the best conceivable bound for class numbers on GRH, $h(-40N) \geq \sqrt{N}$ (in reality such a strong bound is false but our aim here is to show how even such a strong estimate leads to a very large bound for N) so that $|a(N)| \geq \sqrt{N}/4$. Let us also suppose, very charitably, that the Iwaniec-Duke results give $|a(N)| \leq \tau(N)N^{3/7}(\log 2N)^2$. Even with these precise estimates (which are unlikely to be realised in point of act) we require $N \geq (4\tau(N)\log^2(2N))^{14}$ in order to obtain a contradiction. This occurs only if $N \geq 10^{75}$; a bound which is numerically infeasible.

In order to attain a numerically feasible bound we are forced to assume, in addition to the Riemann hypothesis for Dirichlet L-functions, the Riemann hypothesis for the Hasse-Weil L-functions, $L(E(-10N), s)$. Since the Riemann hypothesis for $L(E(-10N), s)$ implies the Lindelöf bound, $|L(E(-10N), 1)| \ll N^{\epsilon}$ (in the language of the preceding passages this is equivalent to the Ramanujan bound for $|a(N)|$), at first glance we may expect a straight-forward, feasible solution to our problem. In practice however the familiar deduction of the Lindelöf bound from the Riemann hypothesis (see Theorem 13.2 of E. C. Titchmarsh [T] for a proof in the case of $\zeta(s)$; the ideas generalize easily) leads, at best, to a bound of the form

$$|L(E(-10N),1)| \leq \exp\left(\frac{3}{2}\frac{\log q}{\log\log q}\right),$$

where $q = 1600N^2$ is the conductor of $E(-10N)$ (see Proposition 2). Even assuming the very strong bound $h(-40N) \geq \sqrt{N}$, we require $N \geq 10^{85}$ before Theorem 2 yields a contradiction.

Thus we are forced to develop a completely different line of attack: one that involves, as a cursory examination of sections 6 through 10 reveals, considerable technical difficulties. The two main weapons in our arsenal are explicit formulae (see Lemmata 1 and 2 of §6) and Hadamard's factorization formula (see Lemma 3 of §6). This contrasts sharply with the traditional methods for deducing the Lindelöf bound from Riemann hypothesis which use tools from complex analysis (namely the Borel-Caratheodory theorem and Hadamard's three circles theorem, see [Ru]) and make no reference to the zeros of the L-function in question. Our strategy is outlined in detail in §6 and using it we succeed in demonstrating that, conditional on GRH, all eligible integers larger than $2 \cdot 10^{10}$ are represented by Ramanujan's form. In view of Galway's numerical verification, mentioned earlier, this provides a conditional answer to Ramanujan's query. One noteworthy feature of our method is that we exploit the fact that both the Hasse-Weil L-function $L(E(-10N),s)$ and the Dirichlet L-function for the number field $\mathbb{Q}(\sqrt{-40N})$ are twists by the same quadratic character $\chi = \left(\frac{-40N}{\cdot}\right)$.

Theorem 3. *Suppose the non-trivial zeros of all Dirichlet L-functions, $L(s,\chi)$, with χ a primitive, real character, have real part $1/2$. Further suppose that the non-trivial zeros of the Hasse-Weil L-functions $L(E(-10N),s)$ (with N a square-free integer coprime to 10) have real part 1. Then the only eligible integers which are not of the form $x^2+y^2+10z^2$ are:*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.$$

Briefly our paper is organized as follows. In §2 we establish some preliminaries and show how to deduce Corollary 1. In sections 3 and 4 we prove Theorems 1 and 2. Section 5 presents some speculations concerning elliptic curves and Ramanujan's form. The remainder of the paper, §6-10, relate to Theorem 3, with §6 containing a detailed plan of the proof.

Acknowledgements

We thank E. Bombieri, W. K. Chan, A. Earnest, A. Granville, J. Hsia, P. Sarnak, I. Kaplansky, and S. Zhang for their help during the preparation of this paper. We especially thank S. Miller and W. Galway for their extensive computations.

2. Preliminary Remarks

Let $r_1(n)$ and $r_2(n)$ denote the number of representations of n by ϕ_1 and ϕ_2 respectively, and define the weight 3/2 cusp form $f(z)=\sum_{n=1}^{\infty}a(n)q^n$, where $q:=e^{2\pi iz}$, by

$$(5)\quad f(z):=\sum_{n=1}^{\infty}a(n)q^n=\frac{1}{4}\sum_{n=1}^{\infty}(r_1(n)-r_2(n))q^n=q-q^3-q^7-q^9+2q^{13}+q^{15}+\dots.$$

This form has the special property that its Shimura lift [Sh] is the weight 2 cusp form

$$(6)\quad F(z)=\eta^2(2z)\eta^2(10z)=\sum_{n=1}^{\infty}A(n)q^n=q-2q^3-q^5+2q^7+q^9+2q^{13}+2q^{15}-6q^{17}-\dots,$$

where $\eta(z):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function. It is important to note that $F(z)$ is the inverse Mellin transform of $L(E,s)$, the Hasse-Weil $L-$function for the elliptic curve given in (3). Consequently by Hasse, it is well known that $|A(p)|\le 2\sqrt{p}$ for every prime p.

Let A_i be the matrices representing the forms ϕ_i respectively. Specifically this means that

$$A_1:=\begin{pmatrix}1&0&0\\0&1&0\\0&0&10\end{pmatrix}\quad\text{and}\quad A_2:=\begin{pmatrix}2&0&-1\\0&2&0\\-1&0&3\end{pmatrix}.$$

A 3×3 matrix B with determinant 1 is called an automorph of A_i if $B^TA_iB=A_i$. Then it is easy to verify that the automorphs of A_1 are the eight matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix},$$

and the four automorphs of A_2 are

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&1\\0&-1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&-1\\0&-1&0\\0&0&-1\end{pmatrix}.$$

Two representations of N by ϕ_i, say (x,y,z) and (x',y',z'), are called *essentially distinct* if there is no automorph B of A_i with the property that

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=B\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Let $G(N)$ denote the number of essentially distinct primitive representations of N by the genus of Ramanujan's form. Observe that if N is square-free, then no two distinct automorphs of representations of N by ϕ_1 (resp. ϕ_2) are equal so that $G(N)=R_1(N)/8+R_2(N)/4$. By [Th.86, Jo1], we obtain the following proposition.

Proposition 1. *If $N > 1$ is a positive eligible integer which is relatively prime to* 10, *then*

$$G(N) = \frac{1}{4}h(-40N).$$

Proposition 2. *If N is a square-free integer coprime to* 10, *then the conductor of* $E(-10N)$ *is* $1600N^2$.

Proof. Let $\Delta(E(-10N))$ denote the discriminant of $E(-10N)$, and let n_p denote the number of irreducible components of the special fibre of the minimal Néron model of $E(-10N)$ at p. If $f_p = \text{ord}_p(\Delta(E(-10N))) + 1 - n_p$, then the conductor of $E(-10N)$ is $\prod_p p^{f_p}$. The integers f_p are easily computed by Tate's algorithm [p. 49, Cr], and they establish that the conductor is $1600N^2$.

□

3. Proof of Theorem 1

The modular form $f(z)$ belongs to $S_{\frac{3}{2}}(40, \chi_{10})$, and is an eigenform of all the half-integral weight Hecke operators $T(p^2)$. Consequently for every prime p, there exists a complex number $\alpha(p)$ such that for every positive integer n

$$\alpha(p)a(n) = a(p^2n) + \chi_{10}(p)\left(\frac{-n}{p}\right)a(n) + \chi_{10}(p^2)pa(n/p^2). \tag{7}$$

Since the image of the Shimura lift of $f(z)$ is the newform $F(z) = \sum_{n=1}^{\infty} A(n)q^n \in S_2(20)$, we find that $\alpha(p) = A(p)$. Since $a(n) = \frac{1}{4}(r_1(n) - r_2(n))$ it follows from (7) that for square-free integers n

$$r_1(np^2) - r_2(np^2) = \left(A(p) - \chi_{10}(p)\left(\frac{-n}{p}\right)\right) \cdot (r_1(n) - r_2(n)). \tag{8}$$

Without loss of generality we may assume that $N > 1$ is a square-free integer where $\gcd(N, 10) = 1$, and $r_1(N) = 0$. Let $p \neq 2, 5$ be prime. If $r_1(Np^2) = 0$, then by (8) we find that

$$\frac{r_2(Np^2)}{r_2(N)} = \left(A(p) - \chi_{10}(p)\left(\frac{-N}{p}\right)\right) \leq A(p) + 1. \tag{9}$$

Since N is square-free, we find, by the definition of primitivity, that

$$r_2(Np^2) = R_2(Np^2) + R_2(N) = R_2(Np^2) + r_2(N).$$

Observe that, since N is square-free, $4G(N) = R_1(N)/2 + R_2(N) = r_2(N)$. Also, since $Np^2 \neq 0$, every primitive essentially distinct representation of Np^2 by ϕ_2 has at least 2 different automorphs whence $2G(Np^2) \leq R_2(Np^2)$. Consequently

$$\frac{r_2(Np^2)}{r_2(N)} = 1 + \frac{R_2(Np^2)}{r_2(N)} \geq 1 + \frac{2G(Np^2)}{4G(N)} = 1 + \frac{G(Np^2)}{2G(N)}. \tag{10}$$

By Proposition 1 and the index formula for $h(-D)$ (see [Co]) it follows that

$$\frac{G(Np^2)}{G(N)} = \frac{h(-40Np^2)}{h(-40N)} = p - \left(\frac{-40N}{p}\right) \geq p - 1$$

which upon substitution in (10) yields

$$\frac{r_2(Np^2)}{r_2(N)} \geq 1 + \frac{G(Np^2)}{2G(N)} \geq \frac{p+1}{2}. \tag{11}$$

From (9) and (11) we conclude that $(p-1)/2 \leq A(p)$ which, in light of Hasse's bound $|A(p)| \leq 2\sqrt{p}$, is impossible for $p \geq 19$. For those primes $p \neq 2, 5$ where $p < 19$, we find that $A(3) = -2, A(7) = 2, A(11) = 0, A(13) = 2$, and $A(17) = -6$, and none of these satisfy $A(p) \geq (p-1)/2$.

We have shown that if N is an eligible square-free integer not represented by ϕ_1 then Np^2 is represented by ϕ_1 for all primes $p \neq 2$, 5. It follows immediately that all eligible non-square-free numbers are represented by ϕ_1. □

4. Proof of Theorem 2

We will derive Theorem 2 as a consequence of a beautiful relation, due to J.-L. Waldspurger [Wal], connecting the Fourier coefficients of half-integer weight cusp forms with the central value of the L-function of their Shimura lift. A special case of his result is stated below.

Theorem. *(Waldspurger) Let $f(z) \in S_{\lambda+\frac{1}{2}}(N, \chi)$ be an eigenform of the Hecke operators T_{p^2} such that its Shimura lift $F(z)$ is a member of $S_{2\lambda}^{\text{new}}(M, \chi^2)$ for an appropriate positive integer M. Denote their respective Fourier expansions by $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ and $F(z) = \sum_{n=1}^{\infty} A(n)q^n$. Let n_1 and n_2 be two positive square-free integers such that $\frac{n_1}{n_2} \in \mathbb{Q}_p^{\times^2}$ for all $p \mid N$. Then*

$$a^2(n_1)L(F, \left(\frac{-1}{\cdot}\right)^{\lambda} \chi^{-1}\chi_{n_2}, \lambda)\chi(n_2/n_1)n_2^{\lambda-\frac{1}{2}} = a^2(n_2)L(F, \left(\frac{-1}{\cdot}\right)^{\lambda} \chi^{-1}\chi_{n_1}, \lambda)n_1^{\lambda-\frac{1}{2}}.$$

Recall from the preceding section that in our case $f(z) \in S_{3/2}(40, \chi_{10})$ and that $f(z)$ is an eigenform of the Hecke operators $T(p^2)$. Further its Shimura lift is an element of $S_2^{\text{new}}(20)$. Thus the hypotheses of Waldspurger's theorem are met.

To apply Waldspurger's theorem we require some knowledge of the square classes modulo 40. It is a simple matter to verify that the set $\mathcal{M} = \{1, 3, 7, 13, 19, 21, 31, 33\}$ contains a representative of all the square classes modulo 40. That is to say for any N coprime to 10 there is an element m of $\mathcal{M}$ with $N/m \in \mathbb{Q}_p^{\times^2}$ for $p = 2$, 5.

Observe that for a square-free eligible integer N, keeping in mind that, since χ_{10} is real, $\chi_{10} = \chi_{10}^{-1}$,

$$\begin{aligned} L(E(-10N), 1) &= \sum_{n=1}^{\infty} \frac{A(n)}{n} \left(\frac{-10N}{n}\right) \\ &= \sum_{n=1}^{\infty} \frac{A(n)}{n} \left(\frac{-1}{n}\right) \chi_{10}^{-1}(n)\chi_N(n) = L(F, \left(\frac{-1}{n}\right)\chi_{10}^{-1}\chi_N, 1). \end{aligned}$$

If m denotes that element of $\mathcal{M}$ which belongs to the same square class as N then, by Waldspurger's theorem and since $\chi_{10}(N/m) = 1$,

$$\frac{a^2(N)}{\sqrt{N}L(E(-10N),1)} = \frac{a^2(m)}{\sqrt{m}L(E(-10m),1)}.$$

Rather pleasantly it follows, from the method discussed in [p.22, Cr], that for all $m \in \mathcal{M}$

$$\frac{a^2(m)}{\sqrt{m}L(E(-10m),1)} = \frac{1}{4\Omega(E(-10))}.$$

We have demonstrated that for a square-free eligible integer N

$$a^2(N) = \frac{1}{16}(R_1(N) - R_2(N))^2 = \frac{\sqrt{N}}{4\Omega(E(-10))}L(E(-10N),1). \tag{12}$$

If $R_1(N) = 0$ then, by Corollary 1, we see that $a(N) = (R_1(N) - R_2(N))/4 = -R_2(N)/4 = -h(-40N)/4$ so that Theorem 2 is an immediate consequence of (12). □

As a consequence, we obtain Corollary 2.

Proof of Corollary 2. If $E(-10N)$ has rank 0, then the Birch and Swinnerton-Dyer Conjecture predicts that

$$L(E(-10N),1) = \frac{\Omega(E(-10N))|\text{Ш}(E(-10N))|}{|E_{\text{tor}(-10N)}|^2} \cdot \prod_p \omega_p(E(-10N)),$$

where $\Omega(E(-10N))$ denotes the real period, $\text{Ш}(E(-10N))$ is the Tate-Shafarevich group, $E_{\text{tor}}(-10N)$ is the torsion subgroup, and $\omega_p(E(-10N))$ is the local Tamagawa number at p, for the elliptic curve $E(-10N)$. Since $E_{\text{tor}}(-10N) \tilde{=} \mathbb{Z}/2\mathbb{Z}$ for all such N, and since $\Omega(E(-10N)) = \frac{\sqrt{N}}{\Omega(E(-10))}$, we obtain Corollary 1.

□

Example 1. In this example we consider the eligible integer $N = 7$, an integer which is not of the form $x^2 + y^2 + 10z^2$. In this case $h(-40N) = h(-280) = 4$, and APECS estimates the L−function value to be $L(E(-280),1) \sim 1.087$. By Theorem 3,

$$h^2(-280) = 16 = \frac{4\sqrt{7}}{\Omega(E(-10))}L(E(-280),1),$$

and APECS estimates this quantity to be ~ 15.996. Regarding Corollary 1, it is conjectured that $|\text{Ш}(E(-280))| = 1$, and it is known that $\prod_p \omega_p(E(-280)) = 16$.

5. Elliptic curves and Ramanujan's form

The methods of this paper apply to many other irregular ternary forms, and work well for inequivalent irregular forms in genera consisting of two classes.

A square-free positive integer N coprime to 10 is called *exceptional* if it satisfies

$$h^2(-40N) = \frac{4\sqrt{N}}{\Omega(E(-10N))} L(E(-10N), 1),$$

and by Theorem 2, every eligible integer N which is not of the form $x^2 + y^2 + 10z^2$ is exceptional. However the converse is false; there exist exceptional N that are of the form $x^2 + y^2 + 10z^2$. It turns out that every exceptional N satisfies either $R_1(N) = 0$ or $R_1(N) = 4R_2(N)$. Moreover the only exceptional $N < 10^7$ for which $R_1(N) = 4R_2(N)$ are $103, 259, 271, 409, 1039$ and 4411 and the proof of Theorem 3 shows (on GRH) that there are no exceptional integers larger than $2 \cdot 10^{10}$.

By the Birch and Swinnerton-Dyer Conjecture (B-SD), these exceptional integers N satisfy

$$h^2(-40N) = \begin{cases} 4^{t(N)+1}|\text{Ш}(E(-10N)| & \text{if } N \neq 409 \\ 64|\text{Ш}(E(-10N))| & \text{if } N = 409, \end{cases}$$

where $t(N)$ denotes the number of odd prime factors of N. However by Gauss' genus theory, the number of genera in $CL(-40N)$ is $2^{t(N)+1}$, and $CL^2(-40N) = \{\alpha^2 \mid \alpha \in CL(-40N)\}$ is a subgroup of $CL(-40N)$ with index $2^{t(N)+1}$. Therefore it seems reasonable to expect that if $N \neq 409$ is an exceptional integer, then

$$\text{Ш}(E(-10N)) \cong CL^2(-40N) \times CL^2(-40N).$$

If we assume the Birch and Swinnerton-Dyer Conjecture, then this assertion is true for the 20 known exceptional integers $N \neq 79, 409, 1039,$ and 2719. For $N = 409$, B-SD predicts $\text{Ш}(E(-10 \cdot 409))$ to be isomorphic to $\mathbb{Z}3 \times \mathbb{Z}3$ which is the odd part of $CL^2(-40 \cdot 409) \times CL^2(-40 \cdot 409)$. For all 24 exceptional N, assuming B-SD we obtain the following:

Ш*(E(-10N))*	*N*
$\mathbb{Z}1 \times \mathbb{Z}1$	$3, 7, 21, 33$
$\mathbb{Z}2 \times \mathbb{Z}2$	$31, 87, 217$
$\mathbb{Z}3 \times \mathbb{Z}3$	$43, 67, 103, 133, 219, 253, 259, 391, 409$
$\mathbb{Z}4 \times \mathbb{Z}4$	79
$\mathbb{Z}5 \times \mathbb{Z}5$	$223, 307, 679$
$\mathbb{Z}6 \times \mathbb{Z}6$	271
$\mathbb{Z}12 \times \mathbb{Z}12$	$1039, 2719$
$\mathbb{Z}2 \times \mathbb{Z}6 \times \mathbb{Z}2 \times \mathbb{Z}6$	4411

S. Zhang computed the 2−torsion in $\text{Ш}(E(-10N))$ when $N = 79, 1039, 2719, 4411$, and with these results we were able to distinguish the conjectured group from $\mathbb{Z}2 \times \mathbb{Z}2 \times \mathbb{Z}2 \times \mathbb{Z}2$ and $\mathbb{Z}2 \times \mathbb{Z}6 \times \mathbb{Z}2 \times \mathbb{Z}6$.

In closing we consider the question of describing those square-free integers N coprime to 10 for which $r_1(N) \neq r_2(N)$. It is easy to see by (12) that $r_1(N) \neq r_2(N)$ if and only if $L(E(-10N), 1) \neq 0$. By Kolyvagin's theorem it follows that if $r_1(N) \neq r_2(N)$ then $E(-10N)$ has rank 0. Conversely if $E(-10N)$ has rank 0 then, assuming B-SD, $L(E(-10N), 1) \neq 0$ so that $r_1(N) \neq r_2(N)$.

6. Preliminaries for the proof of Theorem 3

Suppose $N \geq 2 \cdot 10^{10}$ is an eligible square-free integer which is relatively prime to 10 and is not represented by Ramanujan's form. Let $\chi = \left(\frac{-40N}{\cdot}\right)$ denote the Kronecker-Legendre symbol. For brevity we write

$$L(s) = L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

and

$$L_a(s) = L(E(-10N), s) = \sum_{n=1}^{\infty} \frac{A(n)\chi(n)}{n^s}.$$

Let q be the conductor of $E(-10N)$ so that by Proposition 2, $q = 1600N^2$. It is well-known that $L_a(s)$ satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_a(s) = \pm \left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s) L_a(2-s).$$

We will demonstrate shortly that the sign of the functional equation above may be taken to be positive. Since χ is a primitive character to the modulus $40N = \sqrt{q}$ and since $\chi(-1) = -1$ it follows (see Chapter 12 of [Da2]) that $L(s)$ obeys the functional equation

$$\left(\frac{\sqrt{q}}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L(s) = \frac{i\sqrt{40N}}{\tau(\chi)} \left(\frac{\sqrt{q}}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{2-s}{2}\right) L(1-s)$$

where $\tau(\chi)$ denotes the Gauss sum for the character χ. Since χ is real, we know that $\tau(\chi) = i\sqrt{40N}$ and so the functional equation for $L(s)$ has sign $+1$. Apart from the trivial zeros at $0, -1, -2, \ldots$ our assumption in Theorem 3 ensures that the zeros of $L_a(s)$ lie on the line $\sigma = 1$. Similarly, apart from the trivial zeros at $-1, -3, \ldots,$ the zeros of $L(s)$ are guaranteed by GRH to lie on the line $\sigma = 1/2$. In the sequel θ will denote a complex number, not necessarily the same at each occurence, with $|\theta| \leq 1$.

By Theorem 2 and Dirichlet's class number formula (see [Da2]) we obtain

$$\frac{4\sqrt{N}}{\Omega(E(-10))} L_a(1) = h(-40N)^2 = \left(\frac{\sqrt{40N}L(1)}{\pi}\right)^2 = \frac{40N}{\pi^2} L(1)^2,$$

so that, since $\Omega(E(-10)) \geq 0.7191$ and $q = 1600N^2$,

$$\frac{L_a(1)}{L(1)^2} \geq \frac{7.191\sqrt{N}}{\pi^2} \geq 0.1152 q^{1/4} \geq \frac{2}{7}\left(\frac{q}{4\pi^2}\right)^{1/4}. \tag{13}$$

We see that if the functional equation for $L_a(s)$ had the negative sign then $L_a(1) = 0$, contradicting (13). Thus we may suppose, without loss of generality, that the sign is positive. We prove Theorem 3 by showing that (13) above is not tolerated under the GRH.

To this end we consider

$$F(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^{s-1} \frac{L_a(s)\Gamma(s)}{L(s)L(2-s)}.$$

$F(s)$ is regular in the strip $1/2 < \sigma < 3/2$ and, because of the functional equation of $L_a(s)$, satisfies the functional equation $F(s) = F(2-s)$. Using the Phragmen-Lindelöf principle, see [Ru] for example, to the vertical strip bounded by the lines with real part σ and $2-\sigma$, for some $1 \le \sigma < 3/2$, we see that

$$F(1) = \frac{L_a(1)}{L(1)^2} \le \max_t \max(|F(\sigma+it)|, |F(2-\sigma+it)|) = \max_t |F(\sigma+it)|.$$

For large q the optimal conditional bound for $L_a(1)/L(1)^2$, namely $\ll \exp(A\log q/\log\log q)$ for some positive constant A, is obtained by taking σ very close to 1: roughly $\sigma - 1$ is of the size $1/\log\log q$. However, from the perspective of attaining numerically feasible bounds, this is not very practical. Further in view of the many parameters involved it is desirable to fix, at the outset, a value for σ thereby greatly facilitating the ensuing analysis. We will take $\sigma = 7/6$ and thus concentrate on bounding $|F(7/6+it)|$. While this choice is admittedly somewhat arbitrary we suspect it is not far from optimal (for small q, that is): at any rate, it suffices for our purposes.

Our use of the Phragmen-Lindelöf principle forces us to keep track of the t-aspect of $|F(s)|$. At this juncture we should clarify that this t-dependence, although a nuisance, is quite benign. Indeed we expect that the maximum (over t) is attained at $t = 0$. This is because the Γ-function decays exponentially as t increases while, as will transpire, the other factors constituting $F(s)$ exhibit only a mild polynomial growth in t.

We now proceed to describe our attack on $\max_t |F(7/6+it)|$. Our objective, realized in Propositions 3 and 4, is to obtain an upper bound for $\log|L_a(7/6+it)|$ which consists of a rapidly convergent Dirichlet series along with small error terms, and, similarly, to obtain a lower bound for $\log|L(5/6+it)|$. The first step towards this goal is to obtain explicit formulae for $-L_a'(s)/L_a(s)$ and $-L'(s)/L(s)$.

Recall that, for $\sigma > 1$,

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s}$$

where $\Lambda(n)$, the von Mangoldt function, is $\log p$ if n is a power of the prime p, and 0 otherwise. In Lemma 1 below we will derive an explicit formula for $-L'(s)/L(s)$, expressing it as the sum of a rapidly convergent Dirichlet series, a contribution from the non-trivial zeros of $L(s)$, and two negligible remainder terms.

Lemma 1. *Let X be a positive real number and put*

$$\mathcal{G}_1(s,X) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s} e^{-n/X}.$$

Let ρ denote a generic non-trivial zero of $L(s)$. If $L(s) \ne 0$ then

$$-\frac{L'}{L}(s) = \mathcal{G}_1(s,X) + E_{sig}(s) - \frac{L'}{L}(s-1)X^{-1} - R(s)$$

where

$$E_{sig}(s) = \sum_{\rho} X^{\rho - s}\Gamma(\rho - s),$$

and

$$R(s) = \frac{1}{2\pi i}\int_{-\sigma-1/2-i\infty}^{-\sigma-1/2+i\infty} -\frac{L'}{L}(s+w)\Gamma(w)X^w dw.$$

Proof. By moving the line of integration to the extreme left we see that, for y, $c > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w)y^w dw = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^{-n} = e^{-1/y} \tag{14}$$

whence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s+w)\Gamma(w)X^w dw = \mathcal{G}_1(s, X).$$

We move the line of integration to the line with real part $-\sigma - 1/2$. The pole at $w = 0$ contributes $-L'(s)/L(s)$. The poles at $w = \rho - s$ contribute $-E_{sig}(s)$. The only other pole we encounter is $w = -1$ which contributes $L'(s-1)/(XL(s-1))$.

□

Our next job is to work out the analogue of Lemma 1 for $L_a(s)$. The Euler product for $L_a(s)$ enables us to write the Dirichlet series expansion, for $\sigma > 3/2$,

$$-\frac{L_a'}{L_a}(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s}.$$

We now list some properties of $\lambda(n)$ which will be used often in the sequel. It is easy to check that $\lambda(n) = 0$ unless n is the power of a prime p. Further, if we write $A(p) = \alpha + \overline{\alpha}$ with $|\alpha| = \sqrt{p}$ (this is possible by Hasse's bound) then $\lambda(p^m) = (\alpha^m + \overline{\alpha}^m)\log p$ for all $m \geq 1$. Consequently $|\lambda(p^m)| \leq 2p^{m/2}\log p$ and so $|\lambda(n)| \leq 2\sqrt{n}\Lambda(n)$ for all n.

Lemma 2. *Let $X > 0$ be a real number and put*

$$\mathcal{F}_1(s, X) = \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s} e^{-n/X}.$$

Let ρ_a denote a typical non-trivial zero of $L_a(s)$. If $L_a(s) \neq 0$ then

$$-\frac{L_a'}{L_a}(s) = \mathcal{F}_1(s, X) + R_{sig}(s) + R_{tri}(s) + R_{ins}(s)$$

where

$$R_{sig}(s) = \sum_{\rho_a} X^{\rho_a - s}\Gamma(\rho_a - s), \qquad R_{tri}(s) = \sum_{n=0}^{\infty} X^{-n-s}\Gamma(-n-s),$$

and

$$R_{ins}(s) = \sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!} \frac{L_a'}{L_a}(s-n).$$

Proof. From (14) it follows that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L_a'}{L_a}(s+w)\Gamma(w)X^w dw = \mathcal{F}_1(s,X).$$

We furnish an alternative expression for the LHS of the above identity by moving the line of integration to the far left. The pole at $w=0$ leaves the residue $-L_a'(s)/L_a(s)$. The significant poles at $w=\rho_a - s$ contribute $-R_{sig}(s)$. The trivial poles at $w=-n-s$ for non-negative integers n (arising from the trivial zeros of $L_a(s)$) give $-R_{tri}(s)$. The insignificant poles at $w=-n$ for positive integers n (being the poles of $\Gamma(w)$ apart from 0) yield $-R_{ins}(s)$.

□

Armed with the above two lemmata we proceed to state our desired upper bound for $\log|L_a(7/6+it)|$ and lower bound for $\log|L(5/6+it)|$.

Proposition 3. *For any positive real number X we put*

$$\mathcal{G}(s,X) = \sum_{n=2}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s \log n} e^{-n/X}.$$

Let $s_0 = 5/6+it$ and $s = 7/6+it$. If $X \geq \max(500, 5\log(q/4\pi^2))$ and $\log(q/4\pi^2) \geq 50$ then

$$\log\frac{|L(s_0)|}{|L(s)|} \geq \frac{X(\Re\mathcal{G}(s_0,X) - \Re\mathcal{G}(s,X))}{X-1-7X^{2/3}/9} - \frac{1}{100} - \frac{\log(1+t^2)}{40} - \frac{3}{40X^{1/3}}\log\frac{q}{4\pi^2}.$$

To prove Proposition 3 we begin by integrating both sides of Lemma 1 from $5/6+it$ to $7/6+it$ (obtaining (20) of §7). The two remainder terms, $-\Re\int_{s_0}^{s} R(w)dw$ and $-\Re\int_{s_0}^{s} L'(w-1)/(XL(w-1))dw$, are technically cumbersome to estimate (owing to the t-dependence). However the procedure involved is fairly straight-forward and we see (in §7(i) and (ii)) that their contribution is relatively small. The crux of Proposition 3 lies in our handling of the terms arising from the non-trivial zeros of $L(s)$: namely, $\Re\int_{s_0}^{s} E_{sig}(w)dw$. The contribution of an individual zero $\rho = 1/2+i\gamma$ is

$$\begin{aligned}\int_{s_0}^{s} &\Re X^{\rho-w}\Gamma(\rho-w)dw \\ &\geq -\left|\int_{5/6}^{7/6} X^{1/2-u}\Gamma(1/2-u+i(\gamma-t))du\right| \left(\log\frac{|s-\rho|}{|s_0-\rho|}\right)^{-1}\int_{s_0}^{s}\Re\frac{1}{w-\rho}dw \\ &\geq -\delta(X)\int_{s_0}^{s}\Re\frac{1}{w-\rho}dw,\end{aligned}$$

where

$$\delta(X) = \max_y \left| \int_{1/3}^{2/3} X^{-u}\Gamma(-u+iy)du \right| \left(\frac{1}{2}\log\frac{(y^2+4/9)}{(y^2+1/9)} \right)^{-1}.$$

Note that this maximum exists (indeed, in §7 (iv) we will demonstrate that, for $X \geq 500$, $\delta(X) \leq 7X^{-1/3}/9$) because the exponential decay of $|\Gamma(-u+iy)|$, as $|y|$ increases, swamps the polynomial growth of $(\log((y^2+4/9)/(y^2+1/9)))^{-1}$. We should also point out that the critical ingredient in this argument is the *positivity* of $\Re 1/(w-\rho) = (u-1/2)/((u-1/2)^2+(t-\gamma)^2)$ for $u = \Re w > 1/2$. To resume, we see that our lower bound for the contribution of each individual zero yields, upon summing,

$$\int_{s_0}^{s} \Re E_{sig}(w)dw \geq -\delta(X)\int_{s_0}^{s}\sum_\rho \Re\frac{1}{w-\rho}dw.$$

The key point, at this stage, is to recognize that the logarithmic derivative version of Hadamard's factorization formula (otherwise known as the partial fractions decomposition) enables us to write $\sum_\rho \Re 1/(w-\rho)$ in terms of $\Re L'(w)/L(w)$ and other easily handled functions (see Lemma 3 below). This allows us to estimate (in §7 (iii)) $\int_{s_0}^{s}\sum_\rho \Re 1/(w-\rho)dw$ and by bounding $\delta(X)$, in §7 (iv), we complete the proof of the Proposition.

Proposition 3 may easily be extended to furnish a lower bound for $\log|L(s,\psi)|$ for any primitive character ψ and any $s=\sigma+it$ with $\sigma > 1/2$. In particular one may obtain a lower bound for $|L(1,\psi)|$. While the bound so obtained is asymptotically the same as Littlewood's bound [L] we suspect that the error terms (which were not quantified in [L]) are much better.

We next turn to the analogue of Proposition 3 for $L_a(s)$: namely, obtaining an upper bound for $\log|L_a(7/6+it)|$.

Proposition 4. *Let X be a positive real number and put*

$$\mathcal{F}(s,X) = \sum_{n=2}^{\infty}\frac{\lambda(n)\chi(n)}{n^s\log n}e^{-n/X}.$$

Let $s = 7/6+it$, $s_1 = 11/6+it$ and $s_2 = 27/20+it$. Let

$$\beta(X) = -\frac{7}{20}\frac{X^{7/20}}{\Gamma(13/20)+X^{7/20}}\int_{7/6}^{11/6} X^{1-u}\Gamma(1-u)du$$

and

$$\alpha(X) = \max_y \left| \int_{1/6}^{5/6} X^{-u}\Gamma(-u+iy)du - \frac{\beta(X)}{X^{7/20}}\Gamma(-7/20+iy) \right| \left(\frac{7}{20}+\frac{20}{7}y^2\right).$$

If $X \geq \max(500, 5\log(q/4\pi^2))$ then

$$\log|L_a(s)| \leq \frac{X}{X+1}\Re\mathcal{F}(s,X) + \frac{1}{4} + \frac{\log(1+t^2)}{75} + \frac{(5\alpha(X)-\beta(X))}{4}\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \Re\mathcal{F}_1(s_2,X)\right).$$

Further, if $X \geq \max(5000, 5\log(q/4\pi^2))$ *then*

$$\log|L_a(s)| \leq \frac{X}{X+1}\Re\mathcal{F}(s,X) + \frac{1}{4} + \frac{\log(1+t^2)}{8} + \frac{1}{7X^{1/6}}\log\frac{q}{4\pi^2} - \frac{5}{18X^{1/6}}\Re\mathcal{F}_1(s_2,X).$$

As with Proposition 3, we begin by integrating both sides of Lemma 2 from $s = 7/6+it$ to $s_1 = 11/6+it$ obtaining (29) of §8. The contribution of the trivial zeros of $L_a(s)$ and the insignificant poles of the Γ-function are handled in a straight-forward, albeit tedious, way in §8 (i) and (ii). As in Proposition 3, the heart of the matter lies in our treatment of the contribution of the non-trivial zeros of $L_a(s)$: that is, $\int_s^{s_1} \Re R_{sig}(w)dw$. Let us first discuss the second assertion of Proposition 4. The contribution of an individual zero $\rho_a = 1 + i\gamma_a$ is

$$\begin{aligned}\int_s^{s_1} &\Re X^{\rho_a - w}\Gamma(\rho_a - w)dw \\ &= \theta\left|\int_{7/6}^{11/6} X^{1-u}\Gamma(1-u+i(\gamma_a - t))du\right|\left(\frac{7}{20}+\frac{20}{7}(t-\gamma_a)^2\right)\Re\frac{1}{s_2-\rho_a} \\ &= \theta\gamma(X)\Re\frac{1}{s_2-\rho_a},\end{aligned}$$

where

$$\gamma(X) = \max_y \left|\int_{1/6}^{5/6} X^{-u}\Gamma(-u+iy)du\right|\left(\frac{7}{20}+\frac{20}{7}y^2\right).$$

As with $\delta(X)$ the existence of $\gamma(X)$ is guaranteed by the exponential decay of $|\Gamma(-u+iy)|$ as $|y|$ increases and in §8 (v) we will establish that for $X \geq 5000$, $\gamma(X) \leq 2X^{-1/6}/9$. Summing over all zeros ρ_a we obtain

$$\int_s^{s_1} \Re R_{sig}(w)dw = \theta\gamma(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a}.$$

Notice the crucial role played by the positivity of $\Re 1/(s_2-\rho_a)$ in the above argument. Analogously to Proposition 3, the point is that Hadamard's factorization formula (Lemma 3 below) affords an alternate expression for $\sum_{\rho_a}\Re(s_2-\rho_a)^{-1}$ as a sum of $\Re L_a'(s_2)/L_a(s_2)$ and other easily handled terms. Unlike Proposition 3 the $\Re L_a'(s_2)/L_a(s_2)$ term causes us some difficulties here. We deal with it by using Lemma 2 to essentially reduce the problem to estimating $\Re R_{sig}(s_2)$. This quantitiy is estimated by repeating the argument used above: that is, by bounding each individual term, $\Re X^{\rho_a - s_2}\Gamma(\rho_a - s_2)$, by some function of X times $\Re(s_2-\rho_a)^{-1}$ and then summing and using Hadamard factorization (see estimates (39) through (44) in §8 for more details). Residual traces of these complications may be seen in the presence of the terms involving $\Re\mathcal{F}_1(s_2,X)$ in Proposition 4.

It turns out that the bound obtained in this fashion is not sufficiently effective for 'small' values of q (around e^{50}). The purpose of the first assertion is to obtain more economical constants (at the price of greater complications) for these values of q. Recall from the statement of Proposition 4 the definitions of $\alpha(X)$ and $\beta(X)$. We expect that the maximum over y in the expression defining $\alpha(X)$ is attained at $y = 0$. Supposing this were the case then $\alpha(X)$ would have the value

$$\frac{7}{20}\left|\int_{1/6}^{5/6} X^{-u}\Gamma(-u)du - \frac{\beta(X)}{X^{7/20}}\Gamma(-7/20)\right| = \beta(X).$$

Of course, this is no proof and we have merely demonstrated that $\alpha(X) \geq \beta(X)$ but this expectation should help motivate our definitions of $\alpha(X)$ and $\beta(X)$. In our application (with $X = 500$, in §9) $\alpha(X)$ and $\beta(X)$ will turn out to be very nearly equal. Roughly speaking, we extract further savings by exploiting the fact that if $\log(|L_a(s)|/|L_a(s_1)|) = \int_s^{s_1} -\Re L_a'(w)/L_a(w)dw$ is very large then we would expect $-\Re L_a'(s_2)/L_a(s_2)$ to be very large as well. This works in our favour by forcing (see Lemma 3)

$$\sum_{\rho_a} \Re\frac{1}{s_2-\rho_a} = \left(\frac{1}{2}\log\frac{q}{4\pi^2} + \Re\frac{\Gamma'}{\Gamma}(s_2) + \Re\frac{L_a'}{L_a}(s_2)\right)$$

to be small.

We make this heuristic precise as follows. The contribution of an indvidual zero $\rho_a = 1 + i\gamma_a$ to $\int_s^{s_1} \Re R_{sig}(w)dw$ is, with $y = \gamma_a - t$,

$$\begin{aligned}\Re\int_s^{s_1} X^{\rho_a - w}\Gamma(\rho_a - w)dw &= \Re\beta(X)X^{\rho_a - s_2}\Gamma(\rho_a - s_2)\\ &+ \Re\left(\int_{1/6}^{5/6} X^{-u+iy}\Gamma(-u+iy)du - \beta(X)X^{-7/20+iy}\Gamma(-7/20+iy)\right)\\ &\leq \beta(X)\Re R_{sig}(s_2) + \alpha(X)\Re\frac{1}{s_2-\rho_a}.\end{aligned}$$

Summing over all zeros ρ_a we get

$$\int_s^{s_1} \Re R_{sig}(w)dw \leq \alpha(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a} + \beta(X)\Re R_{sig}(s_2). \tag{15}$$

As usual we use the partial fractions decomposition of Lemma 3 to estimate $\sum_{\rho_a}\Re(s_2-\rho_a)^{-1}$ in terms of $\Re L_a'(s_2)/L_a(s_2)$ and other easy terms and then we use Lemma 2 to reduce the $\Re L_a'(s_2)/L_a(s_2)$ term to $-\Re R_{sig}(s_2)$. In this fashion we loosely obtain

$$\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a} \leq \text{“known terms”} + \Re\frac{L_a'}{L_a}(s_2) \leq \text{“known terms”} - \Re R_{sig}(s_2).$$

Since $\alpha(X)$ and $\beta(X)$ are expected to be nearly equal we see upon using this in (15) that the meddlesome $\Re R_{sig}(s_2)$ term has been practically eliminated! This plan is executed in estimates (38) through (43) of §8 where more details may be found. The net effect of this trick is (loosely) to save a factor of $X^{7/20}/(\Gamma(13/20) + X^{7/20})$ which, although negligible for large X, is of vital importance to the 'small' range of q where we apply it. Proposition 4 may be readily extended to bound any Hecke L-function at a point to the right of the critical line.

We complete our discussion of Proposition 4 by pointing out that our choice of 27/20 for the real part of s_2 is motivated by numerical experiments which indicate that it is close to optimal. These numerics and other computations referred to in the sequel were performed by the authors on a *Silicon Graphics* workstation using *Maple V*.

Propositions 3 and 4 constitute the bulk of our argument for Theorem 3. Using them we establish in §9, without too much difficulty, Theorem 3 for the range $50 \le \log(q/4\pi^2) \le 100$, and in §10 handle the remaining range, $\log(q/4\pi^2) \ge 100$. Since the range $N \le 2\cdot 10^{10}$, where Ramanujan's conjecture has been numerically verified, includes the range $\log(q/4\pi^2) \le 50$ we see that the proof of Theorem 3 is complete.

We conclude this section by proving a few lemmata which will be useful later. We begin with the partial fractions decompositions for $\Re L'/L(s)$ and $\Re L_a'/L_a(s)$.

Lemma 3. *If $L(s) \neq 0$ then*

$$\Re\frac{L'}{L}(s) = -\frac{1}{4}\log\frac{q}{\pi^2} - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right) + \sum_{\rho}\Re\frac{1}{s-\rho}$$

where ρ runs over the non-trivial zeros of $L(s)$. Similarly if $L_a(s) \neq 0$ then

$$\Re\frac{L_a'}{L_a}(s) = -\frac{1}{2}\log\frac{q}{4\pi^2} - \Re\frac{\Gamma'}{\Gamma}(s) + \sum_{\rho_a}\Re\frac{1}{s-\rho_a}$$

where ρ_a runs over the non-trivial zeros of $L_a(s)$.

Proof. Recall that χ is a primitive character to the modulus $40N = \sqrt{q}$. Further since the discriminant, $-40N$, is negative $\chi(-1) = -1$. The first assertion may now be read off from equations (17) and (18) of Davenport [Da2], Chapter 12.

Observe that $(\sqrt{q}/2\pi)^s L_a(s)\Gamma(s)$ is an integral function of order 1 whose zeros are the non-trivial zeros of $L_a(s)$. By Hadamard's factorization formula (see [A]) there exist absolute constants C_1 and C_2 with

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s L_a(s)\Gamma(s) = C_1 e^{C_2 s}\prod_{\rho_a}\left(1-\frac{s}{\rho_a}\right)e^{s/\rho_a}.$$

Upon logarithmic differentiation this yields

$$\log\frac{\sqrt{q}}{2\pi} + \frac{L_a'}{L_a}(s) + \frac{\Gamma'}{\Gamma}(s) = C_2 + \sum_{\rho_a}\left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a}\right).$$

The functional equation reveals that the LHS of the above relation changes sign if we replace s with $2-s$. Since $2-\rho_a$ is a zero of $L_a(s)$ for every non-trivial zero ρ_a we see that $\sum_{\rho_a}\Re(1/(s-\rho_a))$ also changes sign when s is replaced by $2-s$. Thus $\Re C_2 + \sum_{\rho_a}\Re(1/\rho_a)$

must also change sign when we substitute $2-s$ in place of s. Since $\Re C_2 + \sum_{\rho_a} \Re(1/\rho_a)$ is a constant we have established that it must be zero.

□

We also need some inequalities for the logarithmic derivative of the Γ-function. Although the bounds we prove are of a standard, straight-forward nature we know of no convenient reference for them. In establishing these results our priority has been to obtain reasonable estimates, while keeping the effort involved to a minimum, and not to attain the optimal constants possible.

Lemma 4. *Let* $z = x + iy$. *If* $x \geq 1$ *then*

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \frac{11}{3} + \frac{\log(1+x^2)}{2} + \frac{\log(1+y^2)}{2}. \tag{16}$$

Let $<x>= \min|x+n|$ *where the minimum is taken over all non-negative integers* n. *Then*

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \frac{9}{2} + \frac{1}{<x>(1-<x>)} + \log(2+|x|) + \frac{\log(1+y^2)}{2}. \tag{17}$$

Finally if $x > 0$ *then*

$$\Re\frac{\Gamma'}{\Gamma}(z) \leq \frac{\Gamma'}{\Gamma}(x) + \frac{y^2}{x|z|^2} + \log\frac{|z|}{x}. \tag{18}$$

Proof. It is well-known (see [A]) that for complex numbers $w \neq 0, -1, -2, \ldots,$

$$\frac{\Gamma'}{\Gamma}(w) = -\gamma - \frac{1}{w} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+w}\right). \tag{19}$$

Suppose $x \geq 1$. Then, by (19),

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \gamma + \frac{1}{|z|} + \sum_{n=1}^{\infty}\frac{|z|}{n|n+z|} \leq \gamma + \frac{1}{|z|} + \sum_{n=1}^{\infty}\frac{|z|(n+|z|)}{n|n+z|^2}$$
$$\leq \gamma + \frac{1}{|z|} + \sum_{n=1}^{\infty}\left(\frac{|z|}{n^2+|z|^2} + \frac{|z|^2}{n(n^2+|z|^2)}\right).$$

Since $t^2 + |z|^2$ is an increasing function of t,

$$\sum_{n=1}^{\infty}\frac{|z|}{n^2+|z|^2} \leq \int_0^{\infty}\frac{|z|dt}{t^2+|z^2|} = \frac{\pi}{2}$$

and, similarly,

$$\sum_{n=1}^{\infty}\frac{|z|^2}{n(n^2+|z|^2)} \leq \frac{|z|^2}{1+|z|^2} + \int_1^{\infty}\frac{|z|^2dt}{t(t^2+|z|^2)} = \frac{|z|^2}{1+|z|^2} + \frac{\log(1+|z|^2)}{2}.$$

Hence if $x \geq 1$ then

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \frac{11}{3} + \frac{\log(1+|z|^2)}{2} \leq \frac{11}{3} + \frac{\log(1+x^2)}{2} + \frac{\log(1+y^2)}{2},$$

which is (16).

Since (17) follows from (16) if $x \geq 1$, in proving (17) we may suppose that $x < 1$. Let m denote the integer lying between $1-x$ and $2-x$. Then, using (16) for $z+m$,

$$\begin{aligned}\left|\frac{\Gamma'}{\Gamma}(z)\right| &\leq \left|\frac{\Gamma'}{\Gamma}(z+m)\right| + \frac{1}{|z|} + \frac{1}{|z+1|} + \ldots + \frac{1}{|z+m-1|} \\ &\leq \left|\frac{\Gamma'}{\Gamma}(z+m)\right| + \frac{1}{<x>} + \frac{1}{1-<x>} + \log m \\ &\leq \frac{11}{3} + \frac{\log 5}{2} + \frac{1}{<x>(1-<x>)} + \log(2+|x|) + \frac{\log(1+y^2)}{2}\end{aligned}$$

and, since $11/3 + (\log 5)/2 \leq 9/2$, (17) follows.

Using (19) with $w = z$ and $w = x$, subtracting and taking real parts we obtain

$$\begin{aligned}\Re\frac{\Gamma'}{\Gamma}(z) &= \frac{\Gamma'}{\Gamma}(x) + \frac{y^2}{x|z|^2} + \sum_{n=1}^{\infty}\left(\frac{1}{n+x} - \frac{n+x}{|n+z|^2}\right) \\ &= \frac{\Gamma'}{\Gamma}(x) + \frac{y^2}{x|z|^2} + \sum_{n=1}^{\infty}\frac{y^2}{(n+x)|n+z|^2} \\ &\leq \frac{\Gamma'}{\Gamma}(x) + \frac{y^2}{x|z|^2} + \int_0^{\infty}\frac{y^2dt}{(t+x)((t+x)^2+y^2)},\end{aligned}$$

where the final inequality holds since $(t+x)|t+z|^2$ is an increasing function of t. Since

$$\int_0^{\infty}\frac{y^2dt}{(t+x)|t+z|^2} = \int_x^{\infty}\frac{y^2dt}{t(t^2+y^2)} = \log\frac{|z|}{x},$$

we obtain (18).

□

Lastly we require an estimate for the tail of a rapidly convergent series involving prime powers.

Lemma 5. *For positive real numbers X and α we define*

$$\mathcal{H}(\alpha, X) = \sum_{n=1000}^{\infty}\frac{\Lambda(n)}{n^{\alpha}\log n}e^{-n/X}.$$

If $2/3 \leq \alpha < 1$ then

$$\mathcal{H}(\alpha, X) \leq \frac{e^{-1000/X}}{9} + \frac{35e^{-1000/X}}{96}X^{1-\alpha}\Gamma(1-\alpha; 1000/X),$$

where

$$\Gamma(x;y) = \int_y^\infty t^{x-1}e^{-t}dt.$$

Proof. Let $\psi(x) = \sum_{n\le x}\Lambda(n)$. From [RS] we find that for $x \ge 1000$, $\psi(x) \le 16x/15$. Hence

$$\begin{aligned}
\mathcal{H}_1(\alpha, X) &= \int_{1000}^\infty \frac{e^{-t/X}}{t^\alpha \log t} d\psi(t) \\
&\le \frac{e^{-1000/X}}{1000^\alpha \log 1000}(1067 - \psi(1000)) + \frac{16}{15}\int_{1000}^\infty \frac{e^{-t/X}}{t^\alpha \log t}dt \\
&\le \frac{e^{-1000/X}}{9} + \frac{16e^{-1000/X}}{15\log 1000}\int_{1000}^\infty \frac{e^{-u/X}}{u^\alpha}du \\
&\le \frac{e^{-1000/X}}{9} + \frac{35e^{-1000/X}}{96}X^{1-\alpha}\Gamma(1-\alpha; 1000/X),
\end{aligned}$$

as desired.

□

7. Explicit formulae for $L(s)$: Proof of Proposition 3

We integrate both sides of Lemma 1 from $s_0 = 5/6 + it$ to $s = 7/6 + it$. We obtain

$$\log\frac{L(s_0)}{L(s)} = \mathcal{G}(s_0, X) - \mathcal{G}(s, X) + \int_{s_0}^s E_{sig}(w)dw - \int_{s_0}^s R(w)dw + \frac{1}{X}\log\frac{L(s_0 - 1)}{L(s-1)}. \tag{20}$$

(i). The contribution of the remainder term $\int_{s_0}^s R(w)dw$. By the functional equation we see that, with $w = u + it$ and $5/6 \le u \le 7/6$,

$$\begin{aligned}
R(w) = \frac{1}{2\pi i}\int_{-u-1/2-i\infty}^{-u-1/2+i\infty} \Gamma(z)X^z \Bigg(\frac{L'}{L}(1-z-w) + \log\frac{q}{\pi^2} \\
+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2-z-w}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+z+w}{2}\right)\Bigg)dz.
\end{aligned} \tag{21}$$

Since $\chi(2) = 0$,

$$\left|\frac{L'}{L}(1-z-w)\right| \le \sum_{n=1}^\infty \frac{|\Lambda(n)\chi(n)|}{n^{3/2}} \le \left|\frac{\zeta'}{\zeta}\left(\frac{3}{2}\right)\right| - \frac{\log 2}{2^{3/2}} \le \frac{3}{2},$$

and, letting y denote the imaginary part of z, we see by (16) and (17) of Lemma 4 that

$$\begin{aligned}
&\left|\frac{\Gamma'}{\Gamma}\left(\frac{2-z-w}{2}\right)\right| + \left|\frac{\Gamma'}{\Gamma}\left(\frac{1+z+w}{2}\right)\right| \\
&\le \frac{11}{3} + \frac{\log(1+25/16)}{2} + \log(1+(t+y)^2) + \frac{9}{2} + \frac{16}{3} + \log\frac{9}{4} \\
&\le 15 + \log(1+t^2) + \log(1+y^2)
\end{aligned}$$

whence

$$\left|\frac{L'}{L}(1-z-w)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2-z-w}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+z+w}{2}\right)\right| \\ \le 9+\frac{1}{2}\log(1+t^2)+\frac{1}{2}\log(1+y^2). \tag{22}$$

If $z=-u-1/2+iy$ and $u\in[5/6,7/6]$ then, using $|\Gamma(x+iy)|\le|\Gamma(x)|$,

$$|X^z\Gamma(z)| = X^{-u-1/2}\frac{|\Gamma(2+z)|}{|z(1+z)|}\le X^{-u-1/2}\frac{|\Gamma(3/2-u)|}{(1/2+u)(u-1/2)+y^2}.$$

A simple calculation shows that for $X\ge 500$ the RHS of the above inequality attains its maximum, for $u\in[5/6,4/3]$, at $u=5/6$ whence

$$|X^z\Gamma(z)|\le X^{-4/3}\frac{\Gamma(2/3)}{y^2+4/9}\le\frac{3X^{-4/3}}{2(y^2+4/9)}.$$

Using the above inequality along with (22) we see that

$$\frac{1}{2\pi i}\int_{-u-1/2-i\infty}^{-u-1/2+i\infty}X^z\Gamma(z) \\ \times\left(\frac{L'}{L}(1-z-w)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{2-z-w}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+z+w}{2}\right)\right)dz \\ =\frac{3\theta}{4\pi}\int_{-\infty}^{\infty}\frac{X^{-4/3}}{y^2+4/9}\left(9+\frac{1}{2}\log(1+t^2)+\frac{1}{2}\log(1+y^2)\right)dy \\ =\frac{14\theta}{X^{4/3}}+\frac{\theta}{X^{4/3}}\log(1+t^2). \tag{23}$$

Further, if $5/6\le u\le 7/6$ and $X\ge 500$,

$$\frac{1}{2\pi i}\int_{-u-1/2-i\infty}^{-u-1/2+i\infty}\Gamma(z)X^z\log\frac{q}{\pi^2}dz=\log\frac{q}{\pi^2}\sum_{n=2}^{\infty}\frac{(-X)^{-n}}{n!} \\ =\frac{\theta}{X^2}\log\frac{q}{4\pi^2}. \tag{24}$$

Substituting the estimates (23) and (24) into (21) we obtain,

$$R(w)=\frac{14\theta}{X^{4/3}}+\frac{\theta}{X^{4/3}}\log(1+t^2)+\frac{\theta}{X^2}\log\frac{q}{\pi^2}, \tag{25}$$

whence, using $X\ge\max(500,5\log(q/4\pi^2))$,

$$\left|\int_{s_0}^{s}R(w)dw\right|\le\frac{5}{X^{4/3}}+\frac{1}{3X^{4/3}}\log(1+t^2)+\frac{1}{3X^2}\log\frac{q}{\pi^2}\le\frac{1}{200}+\frac{\log(1+t^2)}{1000}. \tag{26}$$

(ii). A lower bound for $\log|L(s_0-1)|/|L(s-1)|$. By the functional equation we see that

$$\log\frac{L(s_0-1)}{L(s-1)} = \log\frac{L(2-s_0)}{L(2-s)} + \frac{1}{3}\log\frac{q}{\pi^2} + \frac{1}{2}\int_{s_0}^{s}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3-w}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right)dw.$$

Taking real parts and noting that, since χ is real, $|L(2-s_0)| = |L(s)|$ and $|L(2-s)| = |L(s_0)|$ we obtain

$$\log\frac{|L(s_0-1)|}{|L(s-1)|} \geq \log\frac{|L(s)|}{|L(s_0)|} + \frac{1}{3}\log\frac{q}{\pi^2} - \frac{1}{2}\int_{s_0}^{s}\left|\frac{\Gamma'}{\Gamma}\left(\frac{3-w}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right||dw|.$$

By (16) and (17) of Lemma 4 we see that

$$\begin{aligned}\left|\frac{\Gamma'}{\Gamma}\left(\frac{3-w}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right| &\leq \frac{11}{3}+\frac{\log(9/4)}{2}+\log(1+t^2)+\frac{9}{2}+\frac{144}{35}+\log 3\\ &\leq 14+\log(1+t^2),\end{aligned}$$

whence, using $\log(q/\pi^2)\geq 50>7$,
(27)

$$\log\frac{|L(s_0-1)|}{|L(s-1)|} \geq \log\frac{|L(s)|}{|L(s_0)|} + \frac{\log(q/\pi^2)}{3} - \frac{7}{3} - \frac{\log(1+t^2)}{6} \geq \log\frac{|L(s)|}{|L_(s_0)|} - \frac{\log(1+t^2)}{6}.$$

(iii). The contribution of the significant zeros: $\Re\int_{s_0}^{s}E_{sig}(w)dw$. Let us recall from the discussion following the statement of Proposition 3 that

$$\Re\int_{s_0}^{s}E_{sig}(w)dw \geq -\delta(X)\sum_{\rho}\int_{s_0}^{s}\Re\frac{1}{w-\rho}dw,$$

where

$$\delta(X) = \max_{y}\left|\int_{1/3}^{2/3}X^{-u}\Gamma(-u+iy)du\right|\left(\frac{1}{2}\log\left(\frac{y^2+4/9}{y^2+1/9}\right)\right)^{-1}.$$

Using Lemma 3, it follows that

$$\begin{aligned}\Re\int_{s_0}^{s}E_{sig}(w)dw &\geq -\delta(X)\int_{s_0}^{s}\left(\frac{1}{4}\log\frac{q}{\pi^2}+\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{w+1}{2}\right)+\Re\frac{L'}{L}(w)\right)dw\\ &= -\delta(X)\left(\frac{1}{12}\log\frac{q}{\pi^2}+\int_{s_0}^{s}\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{w+1}{2}\right)dw+\log\frac{|L(s)|}{|L(s_0)|}\right).\end{aligned}$$

Estimate (18) of Lemma 4 and a little calculation reveal that

$$\begin{aligned}\frac{1}{2}\int_{s_0}^{s}\Re\frac{\Gamma'}{\Gamma}\left(\frac{w+1}{2}\right))dw &\leq \log\frac{\Gamma(13/12)}{\Gamma(11/12)}+\frac{1}{5}\frac{t^2}{(t^2+25/9)}+\frac{1}{12}\log(1+9t^2/25)\\ &\leq \frac{1}{8}\log(1+t^2).\end{aligned}$$

Hence

$$\Re \int_{s_0}^{s} E_{sig}(w)dw \geq -\delta(X)\left(\frac{1}{12}\log\frac{q}{\pi^2} + \frac{\log(1+t^2)}{8} + \log\frac{|L(s)|}{|L(s_0)|}\right). \tag{28}$$

(iv). Bounding $\delta(X)$. We now establish that for $X \geq 500$, $\delta(X) \leq 7/(9X^{1/3})$. Using a computer it is easy to verify that if $y \in (-3,3)$ then

$$\left(\frac{1}{2}\log\left(\frac{y^2+4/9}{y^2+1/9}\right)\right)^{-1}\int_{1/3}^{2/3} X^{-u}|\Gamma(-u+iy)|du$$
$$\leq \left(\frac{X}{500}\right)^{-1/3}\left(\frac{1}{2}\log\left(\frac{y^2+4/9}{y^2+1/9}\right)\right)^{-1}\int_{1/3}^{2/3} 500^{-u}|\Gamma(-u+iy)|du \leq \frac{7}{9X^{1/3}}.$$

Since $|\Gamma(x+iy)|$ decreases as $|y|$ increases we see, with a little calculation, that if $u \in [1/3, 2/3]$ and $|y| \geq 3$ then $|\Gamma(2-u+iy)| \leq |\Gamma(2-u)|/3$. It is also easy to verify that if $|y| \geq 3$ then

$$\left(\log\left(\frac{y^2+4/9}{y^2+1/9}\right)\right)^{-1} \leq \frac{9y^2+1}{2}.$$

Consequently for $|y| \geq 3$

$$\left(\frac{1}{2}\log\left(\frac{y^2+4/9}{y^2+1/9}\right)\right)^{-1}\int_{1/3}^{2/3} X^{-u}|\Gamma(-u+iy)|du$$
$$\leq (9y^2+1)\int_{1/3}^{2/3} X^{-u}\frac{|\Gamma(2-u+iy)|}{(u(1-u)+y^2)}du$$
$$\leq (9y^2+1)\int_{1/3}^{2/3} \frac{X^{-u}}{3}\frac{|\Gamma(2-u)|}{(u(1-u)+y^2)}du$$
$$\leq \left(\frac{X}{500}\right)^{-1/3}\frac{9y^2+1}{3(4y^2+1)}\int_{1/3}^{2/3} 500^{-u}|\Gamma(-u)|du \leq \frac{7}{9X^{1/3}}.$$

As desired we have established that $\delta(X) \leq 7/(9X^{1/3})$.

Using this bound for $\delta(X)$, our assumptions $X \geq 500$ and $\log(q/4\pi^2) \geq 50$, and the estimates (26), (27) and (28) we obtain

$$\log\frac{|L(s_0)|}{|L(s)|} \geq \frac{X(\Re\mathcal{G}(s_0,X) - \Re\mathcal{G}(s,X))}{X-1-7X^{2/3}/9} - \frac{1}{100} - \frac{\log(1+t^2)}{40} - \frac{3}{40X^{1/3}}\log\frac{q}{4\pi^2}.$$

This establishes Proposition 3.

8. Explicit formulae for $L_a(s)$: Proof of Proposition 4

We integrate both sides of Lemma 2 from $s = 7/6+it$ to $s_1 = 11/6+it$. We obtain

$$\log\frac{L_a(s)}{L_a(s_1)} = \mathcal{F}(s,X) - \mathcal{F}(s_1,X) + \int_{s}^{s_1}(R_{sig}(w) + R_{ins}(w) + R_{tri}(w))dw. \tag{29}$$

Using $|\lambda(n)| \le 2\sqrt{n}\Lambda(n)$ and a computer it is easy to check that

$$\log|L_a(s_1)| - \Re\mathcal{F}(s_1, X) \le \sum_{n=2}^{1500} \frac{|\lambda(n)\chi(n)|}{n^{11/6}\log n}(1 - e^{-n/500}) + \sum_{n=1500}^{\infty} \frac{2\Lambda(n)}{n^{4/3}\log n} \le \frac{1}{10}. \tag{30}$$

(i). The contribution of the trivial zeros. Since $|\Gamma(-n-w)| \le |\Gamma(-7/6)|$, if $7/6 \le \Re w \le 11/6$ and $n \ge 0$, and since $X \ge 500$, we see that

$$\left|\int_s^{s_1} R_{tri}(w)dw\right| \le |\Gamma(-7/6)| \sum_{n=0}^{\infty} \int_{7/6}^{11/6} X^{-n-u} du \le \frac{|\Gamma(-7/6)| X^{-1/6}}{(X-1)\log X} \le \frac{1}{500}. \tag{31}$$

(ii). The contribution of the insignificant poles of $\Gamma(z)$. By the functional equation we see that

$$\begin{aligned}\int_s^{s_1} R_{ins}(w)dw &= \sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!} \int_s^{s_1} \frac{L_a'}{L_a}(w-n)dw \\ &= \sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!} \Bigg(\log \frac{L_a(2+n-s_1)}{L_a(2+n-s)} + (s-s_1)\log\frac{q}{4\pi^2} \\ &\qquad - \int_s^{s_1} \left(\frac{\Gamma'}{\Gamma}(2-w+n) + \frac{\Gamma'}{\Gamma}(w-n)\right) dw\Bigg).\end{aligned} \tag{32}$$

Since $X \ge 5\log(q/4\pi^2)$

$$\sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!}(s-s_1)\log\frac{q}{4\pi^2} = \frac{2}{3}\log\frac{q}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} X^{-n} \le \frac{2}{15}. \tag{33}$$

A simple computer calculation reveals that for $n \ge 2$

$$\left|\log\frac{L_a(2+n-s_1)}{L_a(2+n-s)}\right| \le \sum_{m=2}^{\infty} \frac{|\lambda(m)\chi(m)|}{\log m}\left(\frac{1}{m^{13/6}} - \frac{1}{m^3}\right) \le 1$$

and that

$$|\log L_a(3-s)| \le \sum_{m=2}^{\infty} \frac{|\lambda(m)\chi(m)|}{m^{11/6}\log m} \le \frac{9}{10}.$$

Hence, for $X \ge 500$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!} \log\frac{|L_a(2+n-s_1)|}{|L_a(2+n-s)|} &\le -\frac{\log|L_a(3-s_1)|}{X} + \frac{9}{10X} + \sum_{n=2}^{\infty} \frac{X^{-n}}{n!} \\ &\le -\frac{\log|L_a(s)|}{X} + \frac{1}{X};\end{aligned} \tag{34}$$

here we have used the fact that, since χ is a real character, $|L_a(3-s_1)| = |L_a(s)|$. By (16) and (17) of Lemma 4 we see that,

$$\begin{aligned}
&\int_s^{s_1} \left(\frac{\Gamma'}{\Gamma}(2-w+n) + \frac{\Gamma'}{\Gamma}(w-n) \right) dw \\
&= \theta \int_{7/6}^{11/6} \left(\frac{49}{6} + \log((3-u+n)(2+n-u)) + \frac{1}{(u-1)(2-u)} + \log(1+t^2) \right) du \\
&= \frac{2\theta}{3}\Big(\frac{49}{6} + \log(1+t^2) + \log(n+2) + \log(n+1)\Big) + \theta\frac{10}{3} \\
&= \theta\left(9 + \frac{2\log(1+t^2)}{3} + \frac{2\log(n+1)}{3} + \frac{2\log(2+n)}{3}\right).
\end{aligned}$$

Consequently, for $X \geq 500$,

$$-\sum_{n=2}^{\infty} \frac{(-X)^{-n}}{n!} \int_s^{s_1} \left(\frac{\Gamma'}{\Gamma}(2-w+n) + \frac{\Gamma'}{\Gamma}(w-n) \right) dw = \frac{6\theta}{X^2} + \frac{\log(1+t^2)}{2X^2}. \tag{35}$$

Using (18) of Lemma 4 we see that, with $w = u+it$ and $7/6 \leq u \leq 11/6$,

$$\Re\frac{\Gamma'}{\Gamma}(3-w) \leq \frac{\Gamma'}{\Gamma}(3-u) + \frac{t^2}{1+t^2} + \frac{1}{2}\log(1+t^2) \leq \frac{\Gamma'}{\Gamma}(3-u) + \log(1+t^2)$$

while

$$\begin{aligned}
\Re\frac{\Gamma'}{\Gamma}(w-1) &= \Re\frac{\Gamma'}{\Gamma}(w) - \Re\frac{1}{w-1} \leq \Re\frac{\Gamma'}{\Gamma}(w) \\
&\leq \frac{\Gamma'}{\Gamma}(u) + \frac{t^2}{t^2+1} + \frac{1}{2}\log(1+t^2) \leq \frac{\Gamma'}{\Gamma}(u) + \log(1+t^2).
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{1}{X}\int_s^{s_1} \Re\left(\frac{\Gamma'}{\Gamma}(3-w) + \frac{\Gamma'}{\Gamma}(w-1) \right) dw &\leq \frac{2}{X}\log\frac{\Gamma(11/6)}{\Gamma(7/6)} + \frac{4}{3X}\log(1+t^2) \\
&\leq \frac{1}{1000} + \frac{\log(1+t^2)}{100};
\end{aligned}$$

which when combined with (33) through (35) yields

$$\Re\int_s^{s_1} R_{ins}(w)dw \leq \frac{7}{50} + \frac{\log(1+t^2)}{75} - \frac{\log|L_a(s)|}{X}. \tag{36}$$

We have shown, by (30), (31), (32) and (36), that,

$$\log|L_a(s)| \leq \frac{X}{X+1}\Re\mathcal{F}(s,X) + \frac{1}{4} + \frac{\log(1+t^2)}{75} + \frac{X}{X+1}\Re\int_s^{s_1} R_{sig}(w)dw. \tag{37}$$

(iii). The contribution of significant zeros. It remains now to obtain an upper bound for $\Re\int_s^{s_1} R_{sig}(w)dw$. Recalling inequality (15) from the discussion following the statement of Proposition 4 and using Lemma 3 we see that

$$\Re\int_s^{s_1} R_{sig}(w)dw \le \Re\beta(X)R_{sig}(s_2) + \alpha(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a}$$

$$= \Re\beta(X)R_{sig}(s_2) + \alpha(X)\left(\frac{1}{2}\log\frac{q}{4\pi^2} + \Re\frac{\Gamma'}{\Gamma}(s_2) + \Re\frac{L_a'}{L_a}(s_2)\right). \tag{38}$$

We note that by Lemma 2,

$$\frac{L_a'}{L_a}(s_2) = -\mathcal{F}_1(s_2,X) - R_{sig}(s_2) - R_{tri}(s_2) - R_{ins}(s_2). \tag{39}$$

Clearly, for $X \ge 500$,

$$|R_{tri}(s_2)| \le \sum_{n=0}^{\infty}|\Gamma(-7/20)|X^{-n-7/20} \le \frac{1}{100}. \tag{40}$$

Next, using the functional equation,

$$R_{ins}(s_2) = \theta\sum_{n=1}^{\infty}\frac{X^{-n}}{n!}$$
$$\times\left(\left|\frac{L_a'}{L_a}(2+n-s_2) + \frac{\Gamma'}{\Gamma}(s_2-n) + \frac{\Gamma'}{\Gamma}(2-s_2+n)\right| + \log\frac{q}{4\pi^2}\right).$$

It is easy to verify that for $n \ge 1$,

$$\left|\frac{L_a'}{L_a}(2+n-s_2)\right| \le \sum_{m=1}^{\infty}\frac{|\lambda(m)\chi(m)|}{m^{8/5}} \le \frac{21}{5}.$$

From (16) and (17) of Lemma 4 we see that

$$\left|\frac{\Gamma'}{\Gamma}(s_2-n)\right| + \left|\frac{\Gamma'}{\Gamma}(2+n-s_2)\right| \le \frac{28}{3} + \log((n+1)(n+2)) + \log(1+t^2)$$

whence

$$R_{ins}(s_2) = \frac{(X+1)\theta}{X^2}\left(\log(1+t^2) + \log\frac{q}{4\pi^2}\right) + \frac{\theta}{10}. \tag{41}$$

By (18) of Lemma 4 and a little calculation we see that

$$\Re\frac{\Gamma'}{\Gamma}(s_2) \le \frac{\Gamma'}{\Gamma}\left(\frac{27}{20}\right) + \frac{20}{27}\frac{t^2}{t^2+729/400} + \frac{1}{2}\log(1+400t^2/729) \le -\frac{1}{10} + \frac{5}{7}\log(1+t^2)$$

so that, by (40) and (41),

$$\Re\frac{\Gamma'}{\Gamma}(s_2) - \Re R_{tri}(s_2) - \Re R_{ins}(s_2) \le \frac{3}{4}\log(1+t^2) + \frac{1}{100}\log\frac{q}{4\pi^2}. \tag{42}$$

Using (39) and (42) in (38) we see that

$$\begin{aligned}\Re\int_s^{s_1} R_{sig}(w)dw &\le \beta(X)\Re R_{sig}(s_2) \\ &\quad + \alpha(X)\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \Re\mathcal{F}_1(s_2,X) - \Re R_{sig}(s_2)\right) \\ &= (\beta(X)-\alpha(X))\Re R_{sig}(s_2) \\ &\quad + \alpha(X)\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \Re\mathcal{F}_1(s_2,X)\right). \end{aligned}\tag{43}$$

(iv). Bounding $\Re R_{sig}(s_2)$. Observe that if $|y|\ge 3$ then $|\Gamma(33/20+iy)| \le |\Gamma(33/20+3i)| \le \Gamma(33/20)/2$ whence

$$|\Gamma(-7/20+iy)|\left(\frac{7}{20}+\frac{20}{7}y^2\right) \le \frac{|\Gamma(33/20+iy)|}{(91/400+y^2)}\left(\frac{7}{20}+\frac{20}{7}y^2\right) \le \frac{20}{7}\frac{\Gamma(33/20)}{2} \le \frac{7}{4}.$$

Using a computer we checked that the above inequality holds for all $-3\le y\le 3$ as well. Thus we have demonstrated that

$$\max_y |\Gamma(-7/20+iy)|\left(\frac{7}{20}+\frac{20}{7}y^2\right) \le \frac{7}{4}.$$

Consequently we see that, using (39), (42) and $X\ge 500$,

$$\begin{aligned}\Re R_{sig}(s_2) &= \sum_{\rho_a} X^{\rho_a - s_2}\Gamma(\rho_a - s_2) = \frac{7\theta}{4X^{7/20}}\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a} \\ &= \frac{\theta}{5}\left(\frac{1}{2}\log\frac{q}{4\pi^2} + \Re\frac{\Gamma'}{\Gamma}(s_2) + \Re\frac{L_a'}{L_a}(s_2)\right) \\ &= \frac{\theta}{5}\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \mathcal{F}_1(s_2,X) - \Re R_{sig}(s_2)\right).\end{aligned}$$

It follows at once that

$$|\Re R_{sig}(s_2)| \le \frac{1}{4}\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \Re\mathcal{F}_1(s_2,X)\right). \tag{44}$$

The first part of the Proposition follows upon combining (43) and (44) with (37).

To obtain the second assertion we recall from the discussion following the statement of Proposition 4 that

$$\Re\int_s^{s_1} R_{sig}(w)dw = \theta\gamma(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a},$$

where

$$\gamma(X) = \max_y \left(\frac{7}{20} + \frac{20}{7}y^2\right) \int_{1/6}^{5/6} X^{-u} |\Gamma(-u+iy)| du.$$

Using Lemma 3 it follows that

$$\Re \int_s^{s_1} R_{sig}(w) dw = \theta\gamma(X) \left(\frac{1}{2}\log\frac{q}{4\pi^2} + \Re\frac{\Gamma'}{\Gamma}(s_2) + \Re\frac{L_a'}{L_a}(s_2)\right).$$

Employing (39), (42) and (44) in the above we conclude that

$$\Re \int_s^{s_1} R_{sig}(w) dw = \theta\gamma(X) \left(\frac{5}{8}\log\frac{q}{4\pi^2} + \frac{21}{20}\log(1+t^2) - \frac{5}{4}\Re\mathcal{F}_1(s_2, X)\right).$$

(v). Bounding $\gamma(X)$**.** To complete the proof of the Proposition it remains to be shown that if $X \geq 5000$ then $\gamma(X) \leq 2X^{-1/6}/9$. Using a computer it is easy to verify that if $y \in (-2, 2)$ then

$$\begin{aligned}&\left(\frac{7}{20} + \frac{20}{7}y^2\right) \int_{1/6}^{5/6} X^{-u} |\Gamma(-u+iy)| du \\ &\qquad \leq \left(\frac{X}{5000}\right)^{-1/6} \left(\frac{7}{20} + \frac{20}{7}y^2\right) \int_{1/6}^{5/6} 5000^{-u} |\Gamma(-u+iy)| du \leq \frac{2}{9X^{1/6}}.\end{aligned}$$

Since $|\Gamma(x+iy)|$ decreases as $|y|$ increases we see, with a little computation, that if $u \in [1/6, 5/6]$ and $|y| \geq 2$ then $|\Gamma(2-u+iy)| \leq |\Gamma(2-u+2i)| \leq 2|\Gamma(2-u)|/5$. Consequently

$$\begin{aligned}&\left(\frac{7}{20} + \frac{20}{7}y^2\right) \int_{1/6}^{5/6} X^{-u} |\Gamma(-u+iy)| du \\ &\qquad \leq \left(\frac{7}{20} + \frac{20}{7}y^2\right) \int_{1/6}^{5/6} X^{-u} \frac{|\Gamma(2-u+iy)|}{(u(1-u)+y^2)} du \\ &\qquad \leq \int_{1/6}^{5/6} X^{-u} \frac{2\Gamma(2-u)}{5(u(1-u)+y^2)} \left(\frac{7}{20} + \frac{20}{7}y^2\right) du \\ &\qquad \leq \frac{7}{20} \int_{1/6}^{5/6} X^{-u} |\Gamma(-u)| du \leq \frac{2}{9X^{1/6}}.\end{aligned}$$

Our desired bound for $\gamma(X)$ follows and with it the Proposition.

9. Proof of Theorem 3: N relatively small

In this section we deal with those eligible integers N satisfying $50 \leq \log(q/4\pi^2) \leq 100$. We take $X = 500 \geq \max(500, 5\log(q/4\pi^2))$. With the help of a computer we have determined that

$$\beta(X) \geq \frac{37}{26} \frac{500^{-1/6}}{\log 500}, \text{ and } \alpha(X) \leq \frac{47}{33} \frac{500^{-1/6}}{\log 500}.$$

For brevity let us write

$$\frac{X}{X+1}\Re\mathcal{F}(s,X)-\frac{X\Re(\mathcal{G}(s_0,X)-\mathcal{G}(s,X))}{X-1-7X^{2/3}/9}-\frac{5\alpha(X)-\beta(X)}{4}\Re\mathcal{F}_1(s_2,X)$$
$$=\sum_{n=2}^{\infty}\Re\frac{\chi(n)}{n^{it}\log n}v(n;X). \tag{45}$$

Being a linear combination of $\lambda(n)$ and $\Lambda(n)$, $v(n;X)$ is supported only on prime powers. It follows from our bounds for $\alpha(X)$ and $\beta(X)$ that,

$$\frac{5\alpha(X)-\beta(X)}{4}\frac{\log n}{n^{27/20}}\leq\frac{1}{n^{7/6}},$$

for all $n\geq 1000$. Upon recalling that $|\lambda(n)|\leq 2\sqrt{n}\Lambda(n)$ it follows that for $n\geq 1000$

$$|v(n;X)|\leq e^{-n/X}\left(\frac{|\lambda(n)|}{n^{7/6}}+\frac{7}{6}\Lambda(n)\left(\frac{1}{n^{5/6}}-\frac{1}{n^{7/6}}\right)\right)\leq\frac{7\Lambda(n)}{3n^{2/3}}e^{-n/X}.$$

Consequently, by Lemma 5,

$$\sum_{n=1000}^{\infty}\Re\frac{\chi(n)}{n^{it}\log n}v(n;X)\leq\frac{7}{3}\mathcal{H}(2/3,X)$$
$$\leq\frac{35e^{-1000/X}}{96}X^{1/3}\Gamma(1/3;1000/X)+\frac{7e^{-1000/X}}{27}\leq\frac{1}{16}. \tag{46}$$

An easy computer calculation reveals that

$$\log|L(s)|-\sum_{n=2}^{1000}\Re\frac{\chi(n)\Lambda(n)}{n^s\log n}\geq-\log|\zeta(7/6)|+\sum_{n=2}^{1000}\frac{\Lambda(n)}{n^{7/6}\log n}\geq-\frac{9}{52}. \tag{47}$$

It is also easy to verify that

$$\log|\Gamma(s)|+\log(1+t^2)\left(\frac{1}{40}+\frac{1}{75}\right)\leq\log|\Gamma(7/6)|\leq-\frac{3}{40}. \tag{48}$$

From Propositions 3 and 4, and using (45) through (48) above, we obtain

$$\log|F(s)|=\log\frac{|L_a(s)\Gamma(s)|}{|L(s)L(s_0)|}+\frac{1}{12}\log\frac{q}{4\pi^2}$$
$$\leq V(t;X)+\frac{3}{5}+\left(\frac{1}{12}+\frac{51}{100}\frac{(5\alpha(X)-\beta(X))}{4}+\frac{3}{40X^{1/3}}\right)\log\frac{q}{4\pi^2}$$
$$\leq V(t;X)+\frac{3}{5}+\frac{9}{67}\log\frac{q}{4\pi^2}, \tag{49}$$

where

$$V(t;X) = \sum_{n=2}^{1000} \Re \frac{\chi(n)}{n^{it}\log n}\left(v(n;X) - \frac{2\Lambda(n)}{n^{7/6}}\right).$$

Using a computer we have determined the following bounds:

$$\sum_{p=37}^{1000} \Re \frac{\chi(p)}{p^{it}\log p}\left(v(p;X) - \frac{2\log p}{p^{7/6}}\right) \le \sum_{p=37}^{1000} \left|\frac{v(p;X)}{\log p} - \frac{2}{p^{7/6}}\right| \le \frac{9}{5};$$

$$\sum_{\substack{n=p,p^2\\ 31\ge p\ge 11}} \Re \frac{\chi(n)}{n^{it}\log n}\left(v(n;X) - \frac{2\Lambda(n)}{n^{7/6}}\right) \le 1;$$

$$\sum_{j=1}^{3} \Re \frac{\chi(7^j)}{7^{ijt}\log 7^j}\left(v(7^j;X) - \frac{2\log 7}{7^{7j/6}}\right) \le \frac{1}{10};$$

and finally

$$\sum_{j=1}^{6} \Re \frac{\chi(3^j)}{3^{ijt}\log 3^j}\left(v(3^j;X) - \frac{2\log 3}{3^{7j/6}}\right) \le 1.$$

Since $\chi(2) = \chi(5) = 0$ the above inequalities exhaust all the terms counted by $V(t;X)$ and hence yield

$$V(t;X) \le \frac{9}{5} + 1 + \frac{1}{10} = \frac{39}{10}.$$

Inserting this in (49) we have shown that

$$\log|F(s)| \le \frac{9}{2} + \frac{9}{67}\log\frac{q}{4\pi^2}.$$

Since $\log(q/4\pi^2) \ge 50$ this is immediately seen to contradict the lower bound $L_a(1)/L(1)^2 \ge 2(q/4\pi^2)^{1/4}/7$; see (13). We have thus established Theorem 3 in the range $50 \le \log(q/4\pi^2) \le 100$.

10. Proof of Theorem 3: N large

In this section we complete the proof of Theorem 3 by handling all eligible integers N satisfying $\log(q/4\pi^2) \ge 100$. In the range $1000 \ge \log(q/4\pi^2) \ge 100$ we will take $X = 5000$ and in the remaining range we take $X = (\log(q/4\pi^2))^2/8$.

Analogously to (45) we write

$$\begin{aligned}\frac{X}{X+1}\Re\mathcal{F}(s,X) - \frac{X\Re(\mathcal{G}(s_0,X) - \mathcal{G}(s,X))}{X - 1 - 7X^{2/3}/9} - \frac{5}{18X^{1/6}}\Re\mathcal{F}_1(s_2,X)\\ = \sum_{n=2}^{\infty} \Re\frac{\chi(n)}{n^{it}\log n} w(n;X).\end{aligned} \tag{50}$$

Like $v(n;X)$, $w(n;X)$ is non-zero only on prime powers and satisfies, for $n \geq 1000$, the upper bound

$$|w(n;X)| \leq e^{-n/X}\left(\frac{|\lambda(n)|}{n^{7/6}} + \frac{7}{6}\Lambda(n)\left(\frac{1}{n^{5/6}} - \frac{1}{n^{7/6}}\right)\right) \leq \frac{7\Lambda(n)}{3n^{2/3}}e^{-n/X}.$$

By Lemma 5 and our choice of X it follows, with a little computation, that

$$\begin{aligned}\sum_{n=1000}^{\infty} \Re \frac{\chi(n)}{n^{it}\log n} w(n;X) &\leq \frac{7}{3}\mathcal{H}(2/3, X) \leq \frac{35}{96}e^{-1000/X}\left(1 + X^{1/3}\Gamma\left(\frac{1}{3}; \frac{1000}{X}\right)\right) \\ &\leq \frac{7}{128}\log\frac{q}{4\pi^2}. \end{aligned} \tag{51}$$

We recall (47) above that

$$\log|L(s)| - \sum_{n=2}^{1000} \Re \frac{\chi(n)\Lambda(n)}{n^s \log n} \geq -\frac{9}{52}; \tag{52}$$

and that analogously to (48)

$$\log|\Gamma(s)| + \log(1+t^2)\left(\frac{1}{8} + \frac{1}{40}\right) \leq \log|\Gamma(7/6)| \leq -\frac{3}{40}. \tag{53}$$

Combining (50) through (53) with Propositions 3 and 4 we obtain, as in the deduction of (49) above,

$$\begin{aligned}\log|F(s)| &\leq W(t;X) + \frac{107}{200} + \left(\frac{1}{12} + \frac{7}{128} + \frac{1}{7X^{1/6}} + \frac{3}{40X^{1/3}}\right)\log\frac{q}{4\pi^2} \\ &\leq W(t;X) + \frac{107}{200} + \frac{20}{113}\log\frac{q}{4\pi^2}; \end{aligned} \tag{54}$$

where

$$W(t,X) = \sum_{n=2}^{1000} \Re \frac{\chi(n)}{n^{it}\log n}\left(w(n;X) - \frac{2\Lambda(n)}{n^{7/6}}\right).$$

Arguing exactly as in the bound for $V(t;X)$ in §9, we find that $W(t;X) \leq 5$. Using this in (54) we obtain

$$\log|F(s)| \leq \frac{28}{5} + \frac{20}{113}\log\frac{q}{4\pi^2}.$$

Since $\log(q/4\pi^2) \geq 100$ this is easily seen to contradict the lower bound of (13): $L_a(1)/L(1)^2 \geq 2(q/4\pi^2)^{1/4}/7$. The proof of Theorem 3 is complete.

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