A BINOMIAL COEFFICIENT IDENTITY 
ASSOCIATED TO A CONJECTURE OF BEUKERS

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If \( n \) is a positive integer, then let \( A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \), and define integers \( a(n) \) by

\[
\sum_{n=1}^{\infty} a(n) q^n := q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - \ldots.
\]

Beukers conjectured that if \( p \) is an odd prime, then

\[
A \left( \frac{p-1}{2} \right) \equiv a(p) \pmod{p^2}.
\]

In [A-O] it is shown that (1) is implied by the truth of the following identity.

Theorem. If \( n \) is a positive integer, then

\[
\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \frac{1}{2k} + \sum_{i=1}^{n+k} \frac{1}{i} + \sum_{i=1}^{n-k} \frac{1}{i} - 2 \sum_{i=1}^{k} \frac{1}{i} \right\} = 0.
\]

Remark. This identity is easily verified using the WZ method. It holds for all positive \( n \), since it holds for \( n=1,2,3 \) (check!), and since the sequence defined by the sum satisfies a certain (homog.) third order linear recurrence equation. To find the recurrence, and its proof, download the Maple package EKHAD and the Maple program zeilWZP from http://www.math.temple.edu/~zeilberg. Calling the quantity inside the braces \( c(n,k) \), compute the WZ pair \((F,G)\), where \( F = c(n,k+1) - c(n,k) \) and \( G = c(n+1,k) - c(n,k) \). Go into Maple, and type read zeilWZP; zeilWZP(k*(n+k)!**2/k!**4/(n-k)!**2,F,G,k,n,N):

References
