A BINOMIAL COEFFICIENT IDENTITY
ASSOCIATED TO A CONJECTURE OF BEUKERS

Scott Ahlgren, Shalosh B. Ekhad, Ken Ono and Doron Zeilberger

If n is a positive integer, then let $A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$, and define integers $a(n)$ by $\sum_{n=1}^{\infty} a(n)q^n := q \prod_{n=1}^{\infty} (1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - \cdots$. Beukers conjectured that if p is an odd prime, then

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}. \tag{1}$$

In [A-O] it is shown that (1) is implied by the truth of the following identity.

Theorem. *If n is a positive integer, then*

$$\sum_{k=1}^{n} k \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \frac{1}{2k} + \sum_{i=1}^{n+k} \frac{1}{i} + \sum_{i=1}^{n-k} \frac{1}{i} - 2\sum_{i=1}^{k} \frac{1}{i} \right\} = 0.$$

Remark. This identity is easily verified using the WZ method. It holds for all positive n, since it holds for n=1,2,3 (check!), and since the sequence defined by the sum satisfies a certain (homog.) third order linear recurrence equation. To find the recurrence, and its proof, download the Maple package EKHAD and the Maple program zeilWZP from `http://www.math.temple.edu/~zeilberg` . Calling the quantity inside the braces $c(n,k)$, compute the WZ pair (F, G), where $F = c(n, k+1) - c(n, k)$ and $G = c(n+1, k) - c(n, k)$. Go into Maple, and type `read zeilWZP; zeilWZP(k*(n+k)!**2/k!**4/(n-k)!**2,F,G,k,n,N):`

References

[A-O] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences* **(in preparation)**.

[B] F. Beukers, *Another congruence for Apéry numbers*, J. Number Th. **25** (1987), 201-210.

Department of Mathematics, Penn State University, University Park, Pennsylvania 16802
E-mail address: `ahlgren@math.psu.edu`

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: `ekhad@math.temple.edu; http://www.math.temple.edu/ ~ekhad`

Department of Mathematics, Penn State University, University Park, Pennsylvania 16802
E-mail address: `ono@math.psu.edu; http://www.math.psu.edu/ono/`

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: `zeilberg@math.temple.edu; http://www.math.temple.edu/~ zeilberg`

The third author is supported by NSF grant DMS-9508976 and NSA grant MSPR-97Y012. The last author is supported in part by the NSF.