PARTITIONS INTO DISTINCT PARTS AND ELLIPTIC CURVES

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Abstract. Let $Q(N)$ denote the number of partitions of N into distinct parts. If $\omega(k) := \frac{3k^2+k}{2}$, then it is well known that

$$Q(N) + \sum_{k=1}^{\infty} (-1)^k \Big(Q(N - 2\omega(k)) + Q(N - 2\omega(-k)) \Big) = \begin{cases} 1 & \text{if } N = \frac{m(m+1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

In this short note we start with Tunnell's work on the 'congruent number problem' and show that $Q(N)$ often satisfies 'weighted' recurrence type relations. For every N there is a relation for $Q(N)$ which may involve a special value of an elliptic curve L-function.

A positive integer D is called a 'congruent number' if there exists a right triangle with rational sidelengths with area D. Over the centuries there have been many investigations attempting to classify the congruent numbers, but little was known until Tunnell [T] brilliantly applied a tour de force of methods and provided a conditional solution to this problem. It turns out that a square-free integer D is not congruent if the coefficient of q^D in a certain power series is non-zero, and assuming the Birch and Swinnerton-Dyer Conjecture D is congruent if the coefficient of q^D is zero.

In this note we start with Tunnell's work and obtain weighted recurrence formulas for $Q(N)$, the number of partitions into distinct parts. We begin by defining the critical objects. Define integers $b(n)$ by the infinite product

$$\sum_{n=1}^{\infty} b(n) q^n := q \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{8n})^2.$$

This is the Fourier series of a weight two modular form that is an eigenform of the Hecke operators. In particular this implies that (see [Ch. 3 §5, K])

$$b(n)b(m) = b(nm) \tag{1}$$

when $\gcd(n, m) = 1$. If D is a square-free integer, then let $L(E(D), s)$ denote the Dirichlet series

$$L(E(D), s) := \sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right) b(n)}{n^s}. \tag{2}$$

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Define integers $a(n)$ by

$$\sum_{n=1}^{\infty} a(n)q^n := \left(q\prod_{n=1}^{\infty}(1-q^{8n})(1-q^{16n})\right)\cdot\left(\sum_{n=-\infty}^{\infty} q^{2n^2}\right). \tag{3}$$

Using the theory of modular forms, if t is a square-free integer, then the following multiplicative property holds where χ_{-1} is a usual Kronecker-Legendre quadratic character, and μ is the usual Möbius function:

$$a(tm^2) = a(t)\sum_{d\mid m}\chi_{-1}(d)\mu(d)\left(\frac{t}{d}\right)b(m/d). \tag{4}$$

This follows easily from the theory of half-integral weight eigenforms (see [Ch. 4 §3, K]). This identity was also employed in [O]. For convenience if m is a positive integer, then define $c(t,m)$ by

$$c(t,m) := \sum_{d\mid m}\chi_{-1}(d)\mu(d)\left(\frac{t}{d}\right)b(m/d). \tag{5}$$

In particular if t is a square-free integer, then $a(tm^2) = a(t)c(t,m)$.

If D is a positive odd square-free integer, then Tunnell proved [T] that

$$L(E(D),1) = \frac{\Omega\cdot a(D)^2}{4\sqrt{D}} \tag{6}$$

where $\Omega := \int_1^{\infty}(x^3-x)^{-1/2}dx \sim 2.622\ldots$. Tunnell proved that if D is a positive odd square-free integer for which $a(D)\neq 0$, then D is not congruent, and assuming the Birch and Swinnerton-Dyer Conjecture he showed that if $a(D)=0$, then D is congruent.

We begin with some elementary results which are motivated by some observations made by T. Amdeberhan. Throughout the paper $\omega(k)$ shall denote the k^{th} pentagonal number defined by $\omega(k) := \frac{3k^2+k}{2}$.

Theorem 1. *If $p\equiv 3 \pmod 4$ is prime and $8N+1 = p^3N'$ where N' is a positive integer coprime to p, then*

$$Q(N) = -\sum_{k=1}^{\infty}\Big((6k+1)Q(N-2\omega(k)) - (6k-1)Q(N-2\omega(-k))\Big).$$

Theorem 2. *If $p\equiv 3 \pmod 4$ is prime and $8N+3 = p^3N'$ where N' is a positive integer coprime to p, then*

$$Q(N) = -\sum_{k=1}^{\infty}\Big((3k+1)Q(N-2\omega(k)-k) - (3k-1)Q(N-2\omega(-k)+k)\Big).$$

Proof of Theorems 1 and 2. Using the well known identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2},$$

it turns out that

$$\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1+q^{8n})(1+q^{2n})^5(1-q^{2n})^3.$$

Since $\sum_{n=0}^{\infty} Q(n)q^n = \prod_{n=1}^{\infty}(1+q^n)$, by Macdonald's identity [M]

$$q\prod_{n=1}^{\infty}\frac{(1-q^{6n})^5}{(1-q^{3n})^2} = q\prod_{n=1}^{\infty}(1+q^{3n})^5(1-q^{3n})^3 = \sum_{0<n\equiv 1,2 \pmod 6} nq^{n^2} - \sum_{0<n\equiv 4,5 \pmod 6} nq^{n^2}$$

we find that

$$\sum_{n=0}^{\infty} a(8n+1)q^{8n+1} = \left(\sum_{n=0}^{\infty} Q(n)q^{8n+1}\right)\cdot\left(\sum_{n=0}^{\infty}(6n+1)q^{16\omega(n)} - \sum_{n=0}^{\infty}(6n+5)q^{16\omega(-n-1)}\right),$$

and

$$\sum_{n=0}^{\infty} a(8n+3)q^{8n+3} = \left(\sum_{n=0}^{\infty} Q(n)q^{8n+3}\right) \cdot\left(\sum_{n=0}^{\infty}(6n+2)q^{16\omega(n)+8n} - \sum_{n=0}^{\infty}(6n+4)q^{16\omega(-n-1)-8n-8}\right).$$

It suffices to prove that $a(8N+1)=0$ (resp. $a(8N+3)=0$). We may write $8N+1 = p^3N_0m^2$ (resp. $8N+3 = p^3N_0m^2$) for some odd integer m coprime to p and some square-free integer N_0 coprime to p. Therefore by (4), (5), and (6) we find that

$$a(8N+1) = a(pN_0)c(pN_0, pm)$$

(resp. $a(8N+3) = a(pN_0)c(pN_0, pm)$). Since $b(p)=0$, which is easy to see since $b(n)=0$ for every $n \equiv 3 \pmod 4$, it turns out that $c(pN_0, pm) = 0$ by (1) and (5). Therefore $a(8N+1)=0$ (resp. $a(8N+3)=0$).

□

Example 1. Theorems 1 and 2 exhibit infinitely many arithmetic progressions for which the recurrence relations hold. For instance in Theorem 1 let $p=3$ and $N' = 24N+11$. Therefore we find that for every positive integer N that

$$Q(81N+37) = -\sum_{k=1}^{\infty}\Big((6k+1)Q(81N+37-2\omega(k)) - (6k-1)Q(81N+37-2\omega(-k))\Big).$$

Although such relations do not hold in general, one can precisely work out the deviation up to a choice of sign.

Theorem 3. *If $8N+1$ is a positive square-free integer and m is an odd integer, then*

$$Q\left(Nm^2+\frac{m^2-1}{8}\right) = \pm 2c(8N+1,m)\sqrt{\frac{\sqrt{8N+1}\cdot L(E(8N+1),1)}{\Omega}}$$

$$-\sum_{k=1}^{\infty}\left((6k+1)Q\left(Nm^2+\frac{m^2-1}{8}-2\omega(k)\right) - (6k-1)Q\left(Nm^2+\frac{m^2-1}{8}-2\omega(-k)\right)\right).$$

Theorem 4. *If $8N+3$ is a positive square-free integer and m is an odd integer, then*

$$Q\left(Nm^2+\frac{3m^2-3}{8}\right)=\pm c(8N+3,m)\sqrt{\frac{\sqrt{8N+3}\cdot L(E(8N+3),1)}{\Omega}}$$
$$-\sum_{k=1}^{\infty}(3k+1)Q\left(Nm^2+\frac{3m^2-3}{8}-2\omega(k)-k\right)$$
$$+\sum_{k=1}^{\infty}(3k-1)Q\left(Nm^2+\frac{3m^2-3}{8}-2\omega(-k)+k\right).$$

Proofs of Theorem 3 and 4. These results follow immediately from the proof of Theorems 1 and 2, and (4), (5) and (6).

□

Remark 1. The author sees no 'rule' for deducing which sign must be chosen in front of the square-roots in Theorems 3 and 4. Obviously it suffices to find a rule for the signs of the $a(N)'$s. Of course combinatorial interpretations of the $a(N)'$s show that the signs come from taking the difference of two partition functions, but these are not simple functions of N.

Example 2. Here we apply Theorem 3 when $N=4$ and $m=1$. The theorem asserts that

$$Q(4)=\pm 2\sqrt{\frac{\sqrt{33}\cdot L(E(33),1)}{\Omega}}-7Q(0)+5Q(2).$$

Using APECS (MAPLE elliptic curve package) one can easily find that $L(E(33),1)\sim 1.826\ldots$, and so the right hand side of the above equation becomes

$$\pm 2\cdot\sqrt{4.000}-7\cdot 1+5\cdot 1.$$

Since $Q(4)=2$ it is evident that the correct sign is $+$.

Now we apply the celebrated theorem of Coates and Wiles [CW] to deduce many cases for which the L-value does not appear in the recurrences.

Corollary 1. *If $8N+1$ is positive and there exist non-zero rational numbers x and y for which*

$$y^2=x^3-(8N+1)^2x,$$

then for every odd integer m

$$Q\left(Nm^2+\frac{m^2-1}{8}\right)=$$
$$-\sum_{k=1}^{\infty}\left((6k+1)Q\left(Nm^2+\frac{m^2-1}{8}-2\omega(k)\right)-(6k-1)Q\left(Nm^2+\frac{m^2-1}{8}-2\omega(-k)\right)\right).$$

Corollary 2. *If $8N+3$ is positive and there exist non-zero rational numbers x and y for which*

$$y^2 = x^3 - (8N+3)^2 x,$$

then for every odd integer m

$$Q\left(Nm^2 + \frac{3m^2-3}{8}\right) = \\ -\sum_{k=1}^{\infty}(3k+1)Q\left(Nm^2 + \frac{3m^2-3}{8} - 2\omega(k) - k\right) \\ + \sum_{k=1}^{\infty}(3k-1)Q\left(Nm^2 + \frac{3m^2-3}{8} - 2\omega(-k) + k\right).$$

Proofs of Corollary 1 and 2. By the proof of Theorems 1 and 2 it is easy to see that it suffices to prove that $a(8N+1) = 0$ (resp. $a(8N+3) = 0$). However by (6) when $8N+1$ (resp. $8N+3$) is square-free this occurs if and only if $L(E(8N+1), 1) = 0$ (resp. $L(E(8N+3), 1) = 0$).

However if D is odd and square-free, then $L(E(D), s)$ is the Hasse-Weil L-function for $E(D)$. (This is one of the principle facts that Tunnell uses in [T].) Consequently by the celebrated Coates-Wiles theorem [CW], if $L(E(D), 1) \neq 0$, then $E(D)$ has only finitely many points (x, y) with rational coordinates.

Since $E(D)$ has infinitely many points with rational coordinates if there exists a single $(x, y) \in E(D)$ with non-zero rational coordinates, then by the celebrated Coates and Wiles theorem $L(E(D), 1) = 0$ if there exists such an (x, y). This also is an important fact employed in Tunnell's paper [T]. Therefore the result follows when $8N+1$ (resp. $8N+3$) is square-free.

If $y^2 = x^3 - (Dm^2)^2 x$ where D is odd and square-free, then the rational numbers $X = x/m^2$ and $Y = y/m^3$ satisfy

$$Y^2 = X^3 - D^2 X.$$

Therefore by the Coates-Wiles theorem and (6) we find that $a(D) = 0$, and consequently that $a(Dm^2) = 0$ by (4). This implies the result for those cases where $8N+1$ (resp. $8N+3$) is not square-free.

□

Example 3. Here we apply Corollary 2 when $N = 27$ and $m = 1$. It is easy to check that $(x, y) = (-144, 1980)$ satisfies

$$y^2 = x^3 - 219^2 x.$$

Therefore the corollary asserts that

$$Q(27) = -4Q(22) + 2Q(26) - 7Q(11) + 5Q(19) + 8Q(6) = 192.$$

A quick computation verifies that $Q(27) = 192$.

We conclude with more infinite families for which these recurrences involve no L-function values.

Corollary 3. *If* $N(i) := (8i+7)(32i+27)(32i+29)$, *then for every positive integer* i *and every odd integer* m

$$Q\left(\frac{N(i)m^2-1}{8}\right) = -\sum_{k=1}^{\infty}(6k+1)Q\left(\frac{N(i)m^2-1}{8}-2\omega(k)\right) + \sum_{k=1}^{\infty}(6k-1)Q\left(\frac{N(i)m^2-1}{8}-2\omega(-k)\right).$$

Corollary 4. *If* $M(i) := (8i+5)(32i+19)(32i+21)$, *then for every positive integer* i *and every odd integer* m

$$Q\left(\frac{M(i)m^2-3}{8}\right) = -\sum_{k=1}^{\infty}(3k+1)Q\left(\frac{M(i)m^2-3}{8}-2\omega(k)-k\right) + \sum_{k=1}^{\infty}(3k-1)Q\left(\frac{M(i)m^2-3}{8}-2\omega(-k)+k\right).$$

Proofs of Corollaries 3 and 4. By the proof of Corollaries 1 and 2 we may assume that $m = 1$. Corollary 3 follows immediately from Corollary 1 since

$$(x(i), y(i)) = \left(\frac{(32i+28)^4+2(32i+28)^2+1}{16}, \frac{(32i+28)^6-5(32i+28)^4-5(32i+28)^2+1}{64}\right)$$

satisfies

$$y(i)^2 = x(i)^3 - (8i+7)^2(32i+27)^2(32i+29)^2x(i).$$

Corollary 4 follows immediately from Corollary 2 since

$$(x(i), y(i)) = \left(\frac{(32i+20)^4+2(32i+20)^2+1}{16}, \frac{(32i+20)^6-5(32i+20)^4-5(32i+20)^2+1}{64}\right)$$

satisfies

$$y(i)^2 = x(i)^3 - (8i+5)^2(32i+19)^2(32i+21)^2x(i).$$

□

Remark 2. Theorems 1 and 2, and Corollaries 1-4 are recurrence relations which exist since (4) and the Coates and Wiles theorem allow us to deduce that $a(n) = 0$ for suitable n. It would be extremely interesting to obtain combinatorial proofs of Theorems 1 and 2, and Corollaries 1-4 along the lines of various 'involution' type arguments as developed by Franklin, Garsia, Joichi, Milne, Stanton and Zeilberger and others. In particular it would be very interesting to see a combinatorial proof of the fact that

$$a((2n+1)(8n+3)(8n+5)) = 0$$

for every positive integer n.

Remark 3. It is curious to note that if D is an odd square-free integer for which $E(D)$ has only finitely many points, then assuming the Birch and Swinnerton-Dyer Conjecture the order of the Tate-Shafarevich group $\text{Ш}(E(D))$ (see [p. 330,T]) can be formulated in terms of $Q(n)$. Let $\tau(N)$ denote the number of positive divisors of N. If $D \equiv 1 \pmod 8$ is a positive square-free integer, then

$$|\text{Ш}(E(D))| = \left(\frac{Q\left(\frac{D-1}{8}\right) + \sum_{k=1}^{\infty}\left((6k+1)Q\left(\frac{D-1}{8} - 2\omega(k)\right) - (6k-1)Q\left(\frac{D-1}{8} - 2\omega(-k)\right)\right)}{\tau(D)}\right)^2.$$

If $D \equiv 3 \pmod 8$ is a positive square-free integer, then

$$|\text{Ш}(E(D))| = 4 \cdot \left(\frac{Q\left(\frac{D-3}{8}\right) + \sum_{k=1}^{\infty}\left((3k+1)Q\left(\frac{D-3}{8} - 2\omega(k) - k\right) - (3k-1)Q\left(\frac{D-3}{8} - 2\omega(-k) + k\right)\right)}{\tau(D)}\right)^2.$$

In 1993 Granville informed the author that these conjectured orders were not even known to be integers. It would be very interesting if a combinatorial method can prove that these numbers are indeed integers.

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