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The partition function in arithmetic progressions

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1. Introduction and statement of results

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. Euler gave the following generating function for $p(n)$, the number of partitions of an integer n:

$$(1)\quad \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots .$$

Ramanujan observed various surprising congruences for $p(n)$ when n is in certain, very special, arithmetic progressions. For instance, in [Ra, p. xix] Ramanujan proclaims:

> "*I have proved a number of arithmetic properties of p(n)...in particular that*
>
> $$p(5n+4) \equiv 0 \pmod 5,$$
> $$\textit{and} \quad p(7n+5) \equiv 0 \pmod 7.$$
>
> *...I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is*
>
> $$p(11n+6) \equiv 0 \pmod{11}.$$
>
> *There are corresponding properties in which the moduli are powers of* $5, 7$, *or* 11... *It appears that there are no equally simple properties for any moduli involving primes other than these three.*"

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There are now many proofs of these congruences (and their generalizations) in the literature (for instance, see [An, An-G, At, G, G-Ki-St, H-Hu, W]), often involving modular equations and various combinatorial constructions.

Although subsequent works show that there are indeed congruences where the modulus contains prime divisors other than $5, 7,$ and 11, it is still widely believed, as Ramanujan suggested, that "simple" congruence properties are "rare". The quantification of this expectation has remained as one of the open problems in the area.

For instance, there do not seem to be any such congruences modulo 2 or 3. The parity of $p(n)$ seems to be quite random. A widely believed "Folklore Conjecture" asserts that $p(n)$ is 'equally often' even and odd; that is that $p(n)$ is even for $\sim \frac{1}{2}X$ positive integers $n \leq X$. Parkin and Shanks [P-S] conducted the first extensive numerical study, and their evidence strongly supports the "Folklore Conjecture." In the direction of this conjecture, Nicolas, Ruzsa and Sárközy [N-R-S-Se] have proved that

$$\#\{N \leq X \mid p(N) \text{ is even}\} \gg \sqrt{X},$$

$$\#\{M \leq X \mid p(M) \text{ is odd}\} \gg_{\epsilon} \sqrt{X} \cdot \exp\left(-(\log 2 + \epsilon)\frac{\log X}{\log\log X}\right). \tag{2}$$

Subbarao [Su] made the following conjecture on the parity of $p(n)$, for those integers n belonging to any given arithmetic progression:

Conjecture 1. *In every progression $r \pmod t$ there are infinitely many $M \equiv r \pmod t$ for which $p(M)$ is odd, and infinitely many $N \equiv r \pmod t$ for which $p(N)$ is even.*

This conjecture had been proved for every arithmetic progression with modulus t (see [O] for precise references) where

$$t \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 16, 20, 40\},$$

using a variety of elegant combinatorial methods, from the works of Garvan, Kolberg, Hirschhorn, Stanton and Subbarao (Note: This corrects the list of such t that appears in [O]. The author carelessly omitted $t = 6, 8,$ and 20.) In [O] the author went a step further by proving that in any progression $r \pmod t$ there are infinitely many $N \equiv r \pmod t$ for which $p(N)$ is even, and that there are infinitely many $M \equiv r \pmod t$ for which $p(M)$ is odd, provided there is one such M. Furthermore, if there is such an M, then the first one is less than an explicit constant $C_{r,t} < 10^{10}t^7$.

Hence, the "even" case of Conjecture 1 has now been verified for every progression, but the "odd" case remains open. However, we have a simple algorithm to determine the truth of the "odd" case for any given progression $r \pmod t$: Compute $p(M) \pmod 2$ for $M = r, r+t, r+2t, \ldots$ for all such M up to $C_{r,t}$. As soon as we find one odd number we have verified the conjecture. If all these numbers are even, then we have proved that the conjecture is false. Using an efficient version of this algorithm, K. Burrell (Universal Analytics, Inc.) verified the "odd" case of Conjecture 1 for every progression $r \pmod t$ with $t \leq 10^5$.

Recently, Ahlgren [A] and Serre [N-R-S-Se] have quantified the author's results on Conjecture 1 by obtaining estimates for the number of even (resp. odd) values of $p(N)$ for integers N lying in an arithmetic progression. Both Ahlgren and Serre have proved that in every progression $r \pmod t$

$$\#\{N \leq X \mid N \equiv r \pmod t \text{ and } p(N) \text{ is even }\} \gg_{r,t} \sqrt{X}.$$

and Ahlgren proved that

$$\#\{M \leq X \mid M \equiv r \pmod t \text{ and } p(M) \text{ is odd }\} \gg_{r,t} \frac{\sqrt{X}}{\log X},$$

provided there is at least one such M.

Returning to Ramanujan's claim that "simple" congruence properties are rare, we first make the well known observation that his congruences maybe described in a convenient way. If $\ell = 5, 7$ or 11, then for every non-negative integer n

$$p(\ell n + r_\ell) \equiv 0 \pmod \ell \qquad \text{where} \quad r_\ell \equiv 24^{-1} \pmod \ell. \tag{3}$$

In this paper we first consider a refinement of the "odd" case of Conjecture 1. In view of (3), it is natural to attack the following conjecture.

Conjecture 2. *If t is prime, then in every arithmetic progression $24^{-1} \not\equiv r \pmod t$ there are infinitely many $M \equiv r \pmod t$ for which $p(M)$ is odd.*

In view of the author's earlier work, Conjecture 2 was known to be true for every prime $t \leq 10^5$, but not for any other primes. As a consequence of a general theorem that is proved in Sect. 3 (see Theorem 4), we obtain the following result.

Theorem 1. *Conjecture 2 is true for a set of primes with density exceeding $1 - \frac{1}{10^{1500}}$.*

In Sect. 4 we shall prove a general result (see Theorem 5) which goes some way towards quantifying Ramanujan's assertion that "simple" congruence properties are "rare". We consider the following generalization of Conjecture 2.

Conjecture 3. *Let ℓ be prime. If t is prime, then in every progression $24^{-1} \not\equiv r \pmod t$ there are infinitely many $M \equiv r \pmod t$ for which $p(M) \not\equiv 0 \pmod \ell$.*

As an immediate corollary to Theorem 5 we obtain:

Theorem 2. *If ℓ is an odd prime, then Conjecture 3 is true for a set of primes t with density exceeding $1 - \frac{1}{10^{100}}$.*

In Sect. 2 we develop essential preliminaries that are important to the sequel. In particular we prove Theorem 3, the main vehicle for obtaining "non-congruences". In Sect. 3 we first consider the parity of the partition function, and in Sect. 4 we consider the more general case of its reduction modulo odd primes.

2. Preliminaries

In this section we develop the essential preliminaries regarding modular forms (see [K] for background). As usual, if k is a positive integer, then let $S_k(\Gamma_1(N))$ denote the space of weight k cusp forms with respect to the congruence subgroup $\Gamma_1(N)$. Similarly, if ψ is a Dirichlet character modulo N, then let $S_k(N,\psi)$ denote the space of weight k cusp forms with respect to the congruence subgroup $\Gamma_0(N)$ with Nebentypus character ψ. As usual, we shall identify all such modular forms $f(z)$ by their Fourier expansions

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

Here $q := e^{2\pi iz}$ is the uniformizing variable for the point at infinity. We begin with the following well known Lemma:

Lemma 1 [III, Sect. 3, Prop. 17 (b), K]. *If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N,\psi)$ and χ is a Dirichlet character modulo t, then*

$$f_\chi(z) := \sum_{n=1}^{\infty} \chi(n)a(n)q^n \in S_k(Nt^2, \psi\chi^2).$$

Now we recall the Hecke operators. If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N,\psi)$ and $p \nmid N$ is prime, then the Hecke operator $T(p,k,\psi)$ acts on $f(z)$ and returns the cusp form

$$T(p,k,\psi)|f(z) = \sum_{n=1}^{\infty} \left(a(np) + p^{k-1}\psi(p)a(n/p)\right) q^n \in S_k(N,\psi). \tag{4}$$

Here $a(n/p) = 0$ if $p \nmid n$.

Lemma 2. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N,\psi)$, and χ a Dirichlet character modulo t. If $p \nmid Nt^2$ is prime, then $T(p,k,\psi\chi^2)|f_\chi(z) \in S_k(Nt^2,\psi\chi^2)$ and is given by*

$$T(p,k,\psi\chi^2)|f_\chi(z) = \sum_{n=1}^{\infty} \chi(np)\left(a(np) + p^{k-1}\psi(p)a(n/p)\right) q^n.$$

Proof. That $T(p,k,\psi\chi^2)|f_\chi(z) \in S_k(Nt^2,\psi\chi^2)$ is immediate from Lemma 1 and (4). The claim follows immediately by (4) and the definition of $f_\chi(z)$ since

$$T(p,k,\psi\chi^2)|f_\chi(z) := \sum_{n=1}^{\infty} \left(\chi(np)a(np) + p^{k-1}\psi(p)\chi^2(p)\cdot\chi(n/p)a(n/p)\right) q^n.$$

□

Using Dirichlet orthogonality we shall obtain the following result.

Lemma 3. *Suppose that* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \psi)$, *and let* $1 \leq r < t$ *be integers for which* $\gcd(r, t) = 1$. *If* $p \nmid Nt^2$ *is prime, then*

$$F(r, t, p; z) := \sum_{n \equiv rp \pmod{t}} a(np)q^n + p^{k-1}\psi(p) \sum_{n \equiv r \pmod{t}} a(n)q^{np} \in S_k(\Gamma_1(Nt^2)).$$

Proof. Recall Dirichlet's theorem that for every integer n

$$\sum_{\chi \bmod t} \overline{\chi(r)}\chi(n) = \begin{cases} \phi(t) & \text{if } n \equiv r \pmod{t}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

If $p \nmid Nt^2$, then define $F(r, t, p; z)$ by

$$F(r, t, p; z) := \frac{1}{\phi(t)} \cdot \sum_{\chi \bmod t} \overline{\chi(rp^2)} \left(T(p, k, \psi\chi^2) | f_\chi(z) \right).$$

It is easy to see that $F(r, t, p; z) \in S_k(\Gamma_1(Nt^2))$ since each $f_\chi \in S_k(\Gamma_1(Nt^2))$. By Lemma 2 and (5) we find that

$$\begin{aligned} F(r, t, p; z) &:= \frac{1}{\phi(t)} \cdot \sum_{\chi \bmod t} \overline{\chi(rp^2)} \sum_{n=1}^{\infty} \chi(np) \left(a(np) + p^{k-1}\psi(p)a(n/p) \right) q^n \\ &= \frac{1}{\phi(t)} \cdot \sum_{n=1}^{\infty} \sum_{\chi \bmod t} \overline{\chi(rp^2)}\chi(np) \left(a(np) + p^{k-1}\psi(p)a(n/p) \right) q^n \\ &= \sum_{np \equiv rp^2 \pmod{t}} \left(a(np) + p^{k-1}\psi(p)a(n/p) \right) q^n \\ &= \sum_{n \equiv rp \pmod{t}} a(np)q^n + p^{k-1}\psi(p) \sum_{n \equiv r \pmod{t}} a(n)q^{np}. \end{aligned}$$

□

Using a theorem of Sturm and Lemma 3, we obtain the following general theorem guaranteeing the existence of non-zero coefficients in arithmetic progressions.

Theorem 3. *Let* ℓ *be prime. If* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \psi)$ *has integer coefficients, and* $1 \leq r < t$ *are coprime integers for which there is an*

$$n_0 \equiv r \pmod{t} \quad \text{with} \quad a(n_0) \not\equiv 0 \pmod{\ell},$$

then for every sufficiently large prime $p \nmid \ell Nt^2$ *there is an integer* n *for which*

$$n \equiv rp \pmod{t} \quad \text{and} \quad a(np) \not\equiv 0 \pmod{\ell}.$$

Proof. Sturm [Theorem 1, Stu] proved that if $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_k(\Gamma_1(N_g))$ has integer coefficients and has the property that $b(n) \equiv 0 \pmod{\ell}$ for every

$$n \leq C(k, N_g) := \frac{k}{12} \cdot N_g^2 \prod_{p|N_g} \left(1 - \frac{1}{p^2}\right), \tag{6}$$

then $g \equiv 0 \pmod{\ell}$ (i.e. $b(n) \equiv 0 \pmod{\ell}$ for every n).

Suppose that $p \nmid \ell N t^2$ is a prime for which $p > C(k, Nt^2)$, and suppose that there are no integers $n \equiv rp \pmod{t}$ for which $a(np) \not\equiv 0 \pmod{\ell}$. Then by Lemma 3 we find that $F(r,t,p;z) \in S_k(\Gamma_1(Nt^2))$ and

$$F(r,t,p;z) \equiv p^{k-1}\psi(p) \sum_{n \equiv r \pmod{t}} a(n)q^{np} \pmod{\ell}.$$

Hence, $F(r,t,p;z) \not\equiv 0 \pmod{\ell}$ but has the property that the first exponent with non-zero coefficient modulo ℓ is a positive multiple of p. By hypothesis, this is larger than $C(k, Nt^2)$ and therefore contradicts Sturm's theorem (6). This proves that if p is sufficiently large, then there are integers $n \equiv rp \pmod{t}$ for which $a(np) \not\equiv 0 \pmod{\ell}$. This completes the proof. □

3. Application to the parity of the partition function

Now we apply the results of Sect. 2 to the parity of $p(n)$. The Dedekind eta-function is the principal modular form of interest in this paper; it is defined by the infinite product

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n).$$

As usual, let $\Delta(z)$ denote the unique normalized weight 12 cusp form with respect to $SL_2(\mathbb{Z})$. In terms of $\eta(z)$, it is given by

$$\Delta(z) = \eta^{24}(z) = q \prod_{n=1}^{\infty}(1 - q^n)^{24}.$$

The following fundamental fact will be very important (see [O] for details).

Lemma 4 [Proposition 1, O]. *For a given positive integer t, let j be a positive integer for which $2^j > \frac{t}{24}$. Define $f_{t,j}(z)$ by*

$$f_{t,j}(z) := \frac{\eta(24z)}{\eta(48z)} \cdot \Delta^{2^j}(24tz) = \sum_{n=1}^{\infty} a_{t,j}(n)q^n.$$

Then $f_{t,j}(z)$ is in $S_{2^j \cdot 12}(1152t, \chi_2)$, and its Fourier expansion satisfies

$$f_{t,j}(z) = \sum_{n=1}^{\infty} a_{t,j}(n)q^n \equiv \left(\sum_{n=0}^{\infty} p(n)q^{24n-1}\right)\left(\sum_{n=0}^{\infty} q^{2^j \cdot 24t(2n+1)^2}\right) \pmod{2}. \tag{7}$$

Here χ_2 *denotes the usual Kronecker character for* $\mathbb{Q}(\sqrt{2})$.

In view of this Lemma we obtain the following easy Lemma.

Lemma 5. *If* t *is a positive integer, then let* j *be any non-negative integer for which* $2^j > t/24$. *There is an integer* $n \equiv r \pmod{t}$ *for which* $p(n)$ *is odd if and only if there exists an integer* $n' \equiv 24r - 1 \pmod{24t}$ *for which* $a_{t,j}(n')$ *is odd.*

Proof. By (7) it is easy to see that

$$a_{t,j}(n') \equiv \sum_{k=0}^{\infty} p\left(\frac{n' - 2^j \cdot 24t(2k+1)^2 + 1}{24}\right) \pmod{2}.$$

The result now follows easily. □

In view of these preliminary lemmas and Theorem 3 we obtain the following general theorem.

Theorem 4. *Let* $0 \le r < t$ *be integers for which* $\gcd(24r - 1, t) = 1$. *If there exists an integer* $n \equiv r \pmod{t}$ *for which* $p(n)$ *is odd, then for every integer* s *coprime to* $24t$ *there are infinitely many integers* $M \equiv s^2(r - 24^{-1}) + 24^{-1} \pmod{t}$ *for which* $p(M)$ *is odd.*

Proof. Let j be a positive integer for which $2^j > t/24$ and let $f_{t,j}(z)$ be as defined in Lemma 4. By Lemma 5, there is an integer $n_0 \equiv 24r - 1 \pmod{24t}$ for which $a_{t,j}(n_0)$ is odd. By Theorem 3, for every sufficiently large prime p there is an integer $m \equiv (24r - 1)p \pmod{24t}$ for which $a_{t,j}(mp)$ is odd. In particular, for every residue class $s \pmod{24t}$ with $\gcd(s, 24t) = 1$ we find an integer $m \equiv (24r - 1)s^2 \pmod{24t}$ for which $a_{t,j}(m)$ is odd. Therefore, by Lemma 5 again, for each such s there are integers $M \equiv s^2(r - 24^{-1}) + 24^{-1} \pmod{t}$ for which $p(M)$ is odd. Hence by [Main Theorem 2, O], there are infinitely many such M for which $p(M)$ is odd. □

Corollary 1. *If* $t > 3$ *is prime, and there are two integers* n_0 *and* n_1 *for which*

$$(i) \qquad p(n_0) \equiv p(n_1) \equiv 1 \pmod{2},$$
$$(ii) \quad \left(\frac{24n_0 - 1}{t}\right)\left(\frac{24n_1 - 1}{t}\right) = -1,$$

then in every progression $24^{-1} \not\equiv r \pmod{t}$ *there are infinitely many* $M \equiv r \pmod{t}$ *for which* $p(M)$ *is odd.*

Proof. Since t is prime, the only $r \pmod{t}$ for which $\gcd(24r - 1, t) \ne 1$ is the residue class $r \equiv 24^{-1} \pmod{t}$. Since t is prime, it is easy to see that Theorem 4 will cover all the residue classes $24^{-1} \not\equiv r \pmod{t}$. □

We now have an algorithm for proving Conjecture 2 for almost every prime: Given any finite set of integers $n_1, n_2, \ldots n_s$ for which $p(n_i)$ are odd, simply find all the arithmetic progressions of primes for which $\left(\frac{24n_i-1}{t}\right)\left(\frac{24n_j-1}{t}\right) = -1$ for any $1 \leq i, j \leq s$. A MAPLE calculation that computed $p(n) \pmod 2$ for every $n \leq 15000$ yields Theorem 1.

4. The partition function modulo ℓ

In this section we consider the reduction of $p(n)$ modulo odd primes ℓ for those n belonging to an arithmetic progression. We begin with the following important lemma.

Lemma 6. *Let ℓ be an odd prime, and t a positive integer. If j is a positive integer for which $\ell^j > 24t$, then*

$$f(t,\ell,j;z) = \sum_{n=1}^{\infty} a(t,\ell,j;n)q^n := \frac{\eta^{\ell^j}(576tz)}{\eta(24z)} \in S_{\frac{\ell^j-1}{2}}\left(24^3t, \chi_{t,\ell,j}\right)$$

where $\chi_{t,\ell,j}$ is the usual Kronecker character for $\mathbb{Q}\left(\sqrt{(-1)^{\frac{\ell^j-1}{2}} \cdot 6t}\right)$. Moreover, the Fourier expansion of $f(t,\ell,j;z)$ factors modulo ℓ as

$$f(t,\ell,j;z) = \sum_{n=1}^{\infty} a(t,\ell,j;n)q^n \equiv$$

$$\equiv \left(\sum_{n=0}^{\infty} p(n)q^{24n-1}\right)\left(\sum_{k=-\infty}^{\infty} (-1)^k q^{24t\ell^j(6k+1)^2}\right) \pmod{\ell}. \tag{8}$$

Proof. Gordon, Hughes, Ligozat and Newman proved the following well known fact about eta-products. If N is a positive integer, and $f(z) := \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-product for which

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},\ \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},\ \text{and}\ \frac{1}{24}\sum_{\delta|N} \frac{N \cdot \gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta} > 0$$

for every integer $d \mid N$, then $f(z) \in S_k(N,\chi)$ where $k := \frac{1}{2} \cdot \sum_{\delta|N} r_\delta$, $s := \prod_{\delta|N} \delta^{r_\delta}$, and χ is the Kronecker character for $\mathbb{Q}(\sqrt{(-1)^k s})$. That $f(t,\ell,j;z) \in S_{\frac{\ell^j-1}{2}}(24^3t, \chi_{t,\ell,j})$ is now immediate. By combining Euler's Pentagonal Number Theorem [Corollary 1.7, An], that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2+k}{2}},$$

with the fact that $(1-X)^{\ell^j} \equiv 1 - X^{\ell^j} \pmod{\ell}$, we obtain (8) from (1). □

Lemma 7. *Let ℓ be an odd prime, t a positive integer, and j an integer for which $\ell^j > 24t$. There is an integer $n \equiv r \pmod{t}$ for which $p(n) \not\equiv 0 \pmod{\ell}$ if and only if there is an integer $n' \equiv 24r - 1 \pmod{24t}$ for which $a(t,\ell,j;n') \not\equiv 0 \pmod{\ell}$.*

Proof. This follows immediately from (8) since

$$a(t,\ell,j;n') \equiv \sum_{k=-\infty}^{\infty} (-1)^k p\left(\frac{n' - 24t\ell^j(6k+1)^2 + 1}{24}\right) \pmod{\ell}.$$

□

In view of Theorem 3, and Lemmas 6 and 7 we prove:

Theorem 5. *Let $0 \le r < t$ be integers for which $\gcd(24r-1,t) = 1$, and let ℓ be an odd prime. If there is an integer $n \equiv r \pmod{t}$ for which $p(n) \not\equiv 0 \pmod{\ell}$, then for every integer s coprime to $24t$ there are infinitely many $M \equiv s^2(r - 24^{-1}) + 24^{-1} \pmod{t}$ for which $p(M) \not\equiv 0 \pmod{\ell}$.*

Proof. Let j be a positive integer for which $\ell^j > 24t$, and let $f(t,\ell,j;z)$ be as in Lemma 6. By Theorem 3 and Lemma 7, if $\gcd(s,24t) = 1$ and $p \equiv s \pmod{24t}$ is a sufficiently large prime, then there is an n for which $n \equiv (24r-1)p \pmod{24t}$ and $a(t,\ell,j;np) \not\equiv 0 \pmod{\ell}$. For such an n, we see that $np \equiv (24r-1)s^2 \pmod{24t}$. If $F(z)$ is the cusp form (see [Lemma 2,O]) defined by

$$F(z) := \sum_{n \equiv (24r-1)s^2 \pmod{24t}} a(t,\ell,j;n)q^n,$$

then $F(z) \not\equiv 0 \pmod{\ell}$.

Suppose that there are only finitely many integers, say $m_1, m_2, \ldots m_c$, for which

$$\begin{aligned} &(i) \qquad m \equiv s^2(r - 24^{-1}) + 24^{-1} \pmod{t},\\ &(ii) \quad p(m) \not\equiv 0 \pmod{\ell}. \end{aligned}$$

It then follows from Lemma 6 and the proof of Lemma 7 that

$$\begin{aligned} 0 &\not\equiv \sum_{n \equiv (24r-1)s^2 \pmod{24t}} a(t,\ell,j;n)q^n \\ &\equiv \sum_{i=1}^{c} \sum_{k=-\infty}^{\infty} (-1)^k p(m_i) q^{24t\ell^j(6k+1)^2 + 24m_i - 1} \pmod{\ell}. \end{aligned} \tag{9}$$

By a simple generalization of the proof of [Lemma 1,O], one easily shows that there are no modular forms whose q-series expansions satisfy (9). Therefore, there must indeed be infinitely many integers $m \equiv s^2(r - 24^{-1}) + 24^{-1} \pmod{t}$ for which $p(m) \not\equiv 0 \pmod{\ell}$. □

As an immediate corollary we obtain:

Corollary 2. *If ℓ and $t > 3$ are odd primes, and there are two integers n_0 and n_1 for which*

$$(i) \qquad p(n_0)p(n_1) \not\equiv 0 \pmod{\ell},$$
$$(ii) \quad \left(\frac{24n_0-1}{t}\right)\left(\frac{24n_1-1}{t}\right) = -1,$$

then in every arithmetic progression $24^{-1} \not\equiv r \pmod{t}$ there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M) \not\equiv 0 \pmod{\ell}$.

Remark 1. Let ℓ be prime. By Corollaries 1 and 2, since $p(0) = p(1) = 1 \not\equiv 0 \pmod{\ell}$, for any prime $t > 3$ with $\left(\frac{-1}{t}\right)\left(\frac{23}{t}\right) = -1$ every arithmetic progression $24^{-1} \not\equiv r \pmod{t}$ has the property that there are infinitely many $M \equiv r \pmod{t}$ with $p(M) \not\equiv 0 \pmod{\ell}$. This holds for every prime t for which $\left(\frac{t}{23}\right) = -1$.

A MAPLE calculation that computed $p(n)$ for every $n \leq 750$ yields Theorem 2. Almost certainly one can obtain a better density by computing many more values of $p(n) \pmod{\ell}$.

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