ROOK THEORY AND $t-$CORES

Jim Haglund, Ken Ono, and Lawrence Sze

Abstract. If t is a positive integer, then a partition of a non-negative integer n is a $t-$core if none of the hook numbers of the associated Ferrers-Young diagram is a multiple of t. These partitions arise in the representation theory of finite groups and also in the theory of class numbers. We prove that if $t = 2, 3$, or 4, then two different $t-$cores are rook equivalent if and only if they are conjugates. In the special case when $t = 4$, since $c_4(n) = \frac{1}{2}h(-32n-20)$ when $8n + 5$ is square-free, the above result suggests a new method of approaching Gauss' class number problem for these discriminants. Unlike the cases where $2 \leq t \leq 4$, it turns out that when $t \geq 5$ there are distinct rook equivalent $t-$cores which are not conjugates. In fact, we conjecture that for all such t, there exists a constant $N(t)$ for which every integer $n \geq N(t)$ has the property that there exists a pair of distinct rook equivalent $t-$cores of n which are not conjugates.

1. Introduction

A *Ferrers board* is a subset of an $N \times N$ chessboard of squares whose rows are non-increasing in length. The number of Ferrers boards with n squares is $p(n)$, the number of partitions of n. The squares of a Ferrers board are labelled with coordinates (i, j) as we would label the entries in a matrix.

Rooks are formal objects which are placed on the squares of a Ferrers board. A *legal placement* of k rooks on a Ferrers board is any placement of k rooks (one per square) with the property that no two rooks are in a common row or column. If k exceeds the number of rows or columns in a Ferrers board, then there is no such legal placement. Early applications of this notion were investigated by Kaplansky and Riordian in [8].

If B is a Ferrers board, then let $r_k(B)$ be the number of legal placements of k rooks on B.

Definition 1. *Two Ferrers boards B_1 and B_2 are* **rook equivalent** *if $r_k(B_1) = r_k(B_2)$ for every positive integer k.*

In particular if B_1 and B_2 are rook equivalent, then since $r_1(B)$ is the number of squares on a board B, it follows that B_1 and B_2 have the same number of squares. By the work of Foata and Schützenberger [4], every rook equivalence class of Ferrers boards contains a unique decreasing Ferrers board, a board with the property that no two rows have the same length. Consequently, it is easy to see that the number of rook equivalence classes of Ferrers boards of size n is $q(n)$, the number of partitions of n into distinct parts. For more on the notion of rook equivalence, see [4,6,11].

1991 *Mathematics Subject Classification.* Primary 05A15; Secondary 11P81.

The second author is supported by NSF grants DMS-9508976 and DMS-9304580.

Now we switch to the language of partitions. If $\Lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ is a partition of n, then the *Ferrers-Young diagram* of Λ is the $s-$rowed collection of nodes:

$$\begin{array}{llllll} \bullet & \bullet & \dots & \bullet & \bullet & \quad \lambda_1 \text{ nodes} \\ \bullet & \bullet & \dots & \bullet & & \quad \lambda_2 \text{ nodes} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \bullet & \dots & \bullet & & & \quad \lambda_s \text{ nodes.} \end{array}$$

We label the nodes in the Ferrers-Young diagram of a partition as we would a matrix. Then the *hook number* $H(i,j)$ of the (i,j) node is one more than the number of nodes directly to the right or directly below the node itself. Alternatively, let λ_j' denote the number of nodes in column j. Then the hook number $H(i,j)$ is defined by

$$H(i,j) := (\lambda_i - i) + (\lambda_j' - j) + 1.$$

Definition 2. *If t is a positive integer, then a partition of n is called a* t**-core** *of n if none of the hook numbers of its associated Ferrers-Young diagram are multiples of t. Moreover, let $c_t(n)$ denote the number of $t-$core partitions of n.*

Remark 1. It is an easy exercise to verify that $c_2(n)$ is given by

$$c_2(n) = \begin{cases} 1 & \text{if } \ n = \frac{m(m+1)}{2} \ \text{ for some integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the only $2-$cores are those partitions whose Ferrers-Young diagram is triangular.

These $t-$core partitions arise in a number of settings. In combinatorial number theory, Garvan, Kim, and Stanton [5] used them to obtain combinatorial proofs of certain special cases of the Ramanujan congruences for $p(n)$. In representation theory $t-$cores, for t prime, first arose in connection with Nakayama's conjecture which describes the distribution of characters of the symmetric group into Brauer blocks. Recently Fong and Srinivasan showed how these partitions arise again in a similar context. They proved that $t-$cores, even when t is composite, describe the distribution of characters of finite general linear groups and unitary groups into Brauer blocks.

When $t = 4$, it turns out that these $t-$cores are important in algebraic number theory. It is shown in [10] that if $8n + 5$ is square-free, then

$$c_4(n) = \frac{1}{2} h(-32n - 20),$$

where $h(-D)$ is the class number of discriminant $-D$ binary quadratic forms. Moreover, it is shown how to construct binary quadratic forms from $4-$cores.

The current investigation may shed some light on Gauss' class number problem [2], that if $h(-D)$ is the class number of binary quadratic forms with discriminant $-D$, then

$$h(-D) \to +\infty$$

as $D \to +\infty$. Although it is too difficult to explicitly construct binary quadratic forms in a way which shows that $h(-D) \to +\infty$, Gauss' problem was solved by Siegel, who proved that for every $\epsilon > 0$ there exists a constant $c(\epsilon)$ for which

$$h(-D) > c(\epsilon)|D|^{\frac{1}{2} - \epsilon}.$$

Unfortunately, the constant $c(\epsilon)$ is inexplicit and depends on the zeros of Dirichlet $L-$functions. In fact the complete classification of discriminants for which $h(-D) = 1$ or 2 was not resolved until late 1960's by Baker and Stark.

By connecting the problem to rank 3 elliptic curves, Goldfeld, Gross, and Zagier, with the aid of intricate computations by Oesterlé, have shown for negative fundamental discriminants D, that

$$h(-D) > \frac{1}{7000} \prod_{p|D} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right) \log|D|$$

where $[x]$ is the greatest integer bracket function. Although this is a very strong unconditional lower bound, it is far from the true order of magnitude which is given by Siegel's theorem. In particular, using the Goldfeld-Gross Zagier result to classify all the discriminants with small class number h is still an extremely difficult computation. Therefore there is interest in developing new methods of interpreting, and hence attacking Gauss' problem.

Since $t-$cores occur in various settings, it is of interest to investigate their structure, and natural relations they may satisfy. In this paper, we examine the rook theory of $t-$cores and show that for $t = 2, 3$, or 4 the rook theory is particularly simple. We begin with some data which illustrate our main result. Define the combinatorial functions $sc_t(n), nsc_t(n)$, and $a_t(n)$ by:

$$\begin{aligned} sc_t(n) &:= \text{number of self-conjugate } \ t - \text{cores of } n, \\ nsc_t(n) &:= \text{number of non-self-conjugate } t - \text{cores of } n, \\ a_t(n) &:= \text{number of rook equivalence classes of } \ t - \text{cores of } n, \end{aligned}$$

where two partitions are *conjugate* if the set of row sizes of one is equal to the set of column sizes of the other. If $t = 3$, then the first few terms of their generating functions are:

$$\begin{aligned} &\sum_{n=0}^{\infty} sc_3(n)q^n = 1 + q + q^5 + q^8 + q^{16} + \dots, \\ &\sum_{n=0}^{\infty} nsc_3(n)q^n = 2q^2 + 2q^4 + 2q^6 + 2q^9 + 2q^{10} + 2q^{12} + 2q^{14} + 2q^{16} + \dots, \\ &\sum_{n=0}^{\infty} a_3(n)q^n = 1 + q + q^2 + q^4 + q^5 + q^6 + q^8 + q^9 + q^{10} + q^{12} + q^{14} + 2q^{16} + \dots . \end{aligned}$$

If $t = 4$, then the first few terms of the generating functions are:

$$\begin{aligned} &\sum_{n=0}^{\infty} sc_4(n)q^n = 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^{10} + q^{12} + \dots, \\ &\sum_{n=0}^{\infty} nsc_4(n)q^n = 2q^2 + 2q^3 + 2q^5 + 2q^6 + 2q^7 + 4q^8 + 4q^9 + 2q^{11} + 6q^{12} + \dots, \\ &\sum_{n=0}^{\infty} a_4(n)q^n = 1 + q + q^2 + 2q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + 2q^8 + 2q^9 + 2q^{10} + q^{11} + 4q^{12} + \dots . \end{aligned}$$

These data suggest that if $t = 3$ or 4, then

$$a_t(n) = \frac{1}{2} nsc_t(n) + sc_t(n).$$

Of course when $t = 2$ the above equality holds trivially. This observation is true and easily follows from the main result of this paper.

Main Theorem. *If $t = 2, 3,$ or 4, then two distinct $t-$cores are rook equivalent if and only if they are conjugates.*

If $t \geq 5$, it is easy to prove that there are distinct non-conjugate $t-$cores which are rook equivalent, and this is proved in Theorem 6. However, more appears to be true.

Conjecture. *If $t \geq 5$, then there exists a constant $N(t)$ with the property that if $n \geq N(t)$, then there exist two distinct rook equivalent $t-$cores of size n which are not conjugates.*

In section 2 we give structure theorems which describe the parts of any 3−core and 4−core, and we also deduce necessary and sufficient conditions for 3−cores and 4−cores to be conjugates. These results follow from the theory of *abaci.* Then in section 3 we deduce the Main Theorem; basically this is accomplished by applying the method of Goldman, Joichi, and White. In section 4 we conclude with a more detailed investigation of the rook equivalence classes which contain $t-$cores with $t = 2, 3,$ or 4. Specifically we compute the size of these classes, and also determine the unique decreasing Ferrers board in each class.

2. Preliminaries

Let $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_s > 0$ be a sequence of non-increasing positive integers that partition a positive integer n. Then for any positive integer $t \geq 2$, there exists an associated *abacus* consisting of s beads on 'rods,' numbered $0, 1, \ldots, t-1$ and infinitely many rows numbered $1, 2, \ldots$ *ad infinitum.* To determine the positions of these beads, first define *structure numbers* B_i by

$$B_i = \lambda_i - i + s. \tag{1}$$

Note that the integers B_i are strictly decreasing by construction. To each B_i, there is a unique pair of integers (r_i, r'_i) where $r_i > 0$ and $0 \leq r'_i \leq t-1$ so that

$$B_i = t(r_i - 1) + r'_i. \tag{2}$$

For each B_i place a bead in position (r_i, r'_i), row r_i and column r'_i.

Example 1. Let $t = 4$ and consider the partition of 13 given by the following Ferrers-Young diagram:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & \\
\bullet & & & & \\
\bullet & & & &
\end{array} .$$

Since the parts are given by $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 3, \lambda_4 = 1$, and $\lambda_5 = 1$, we find that $B_1 = 9, B_2 = 6, B_3 = 5, B_4 = 2$, and $B_5 = 1$. Consequently it is easy to verify that the beads on this abacus are in positions $(3, 1), (2, 2), (2, 1), (1, 2)$ and $(1, 1)$. Graphically, the abacus for this partition is

$$\begin{array}{ccccc}
 & 0 & 1 & 2 & 3 \\
1 & & B_5 & B_4 & \\
2 & & B_3 & B_2 & \\
3 & & B_1 & &
\end{array} .$$

The following fundamental theorem is well known [3,7,9]:

Theorem 1. *Let $\mathfrak{A}$ be an abacus for a partition Λ, and let n_i denote the number of beads in column i. Then Λ is a $t-$core partition if and only if for every $0 \leq i \leq t-1$, the n_i beads in column i are the beads in positions*

$$(1,i),(2,i),\ldots,(n_i,i).$$

In other words, there are no *gaps* between consecutive beads in any column. Furthermore, the top bead in any non-trivial column is in row 1.

Therefore, we let $t-$tuples of non-negative integers $\mathfrak{A}=(n_0,n_1,\ldots,n_{t-1})$ denote the abaci of $t-$cores. Unfortunately, the following well known lemma [3,7,9] shows that abaci do not represent $t-$cores uniquely if we allow for parts of size zero in our partitions.

Lemma 1. *The abaci $\mathfrak{A}_1=(n_0,n_1,\ldots,n_{t-1})$ and $\mathfrak{A}_2=(n_{t-1}+1,n_0,n_1,\ldots,n_{t-2})$ represent the same $t-$core partition.*

Since it is our goal to use abaci as *labels* for all $t-$cores, it is important to normalize the abaci properly. Every $t-$core has a representation by a $t-$tuple, which, by repeated application of Lemma 1, is representable by an abacus of the form $(0,n_1,n_2,\ldots,n_{t-1})$. The bead in the upper left-hand corner of such an abacus corresponds to the smallest part of the partition. The size of this smallest partition part is one of $1,2,\ldots$ or $t-1$ since these are the only integers represented by beads in positions $(1,1),(1,2),\ldots$ or $(1,t-1)$. Since the smallest part in any $t-$core is less than t (otherwise we would have a $t-$hook in the bottom row), it is clear that there is a unique abacus of the form $(0,n_1,n_2,\ldots,n_{t-1})$ for every $t-$core. Hence there is a one to one correspondence between the set of abaci of the form $(0,n_1,n_2,\ldots,n_{t-1})$ and the set of all $t-$cores

$$(0,n_1,n_2,\ldots,n_{t-1}) \longleftrightarrow \{\text{all } t-\text{core partitions}\}$$

where n_i are non-negative integers.

Therefore, throughout we shall assume that the first column in every abacus contains no beads. The following lemma [3] is critical to our study of $t-$cores.

Lemma 2. *If $\mathfrak{A}_1=(0,n_1,n_2,\ldots,n_{t-1})$ is a $t-$core partition of n, then the abacus*

$$\mathfrak{A}_2=(0,n_{t-1}+1,n_1,n_2,\ldots,n_{t-2})$$

represents a $t-$core partition of $n+n_{t-1}+1$.

By Lemma 2, it is easy to see how to trace $t-$cores back to a unique $t-$core of *small size*, $t-$cores which we call *new.*

Definition 3. *If $t \geq 2$, then a* **new** *$t-$core is any partition represented by an abacus of the form*

$$\mathfrak{A}=(0,0,n_2,\ldots,n_{t-1}).$$

Consequently, we find that there are three types of $4-$cores and two types of $3-$cores. Their essential characteristics are captured by the following definitions.

Definition 4. *Let $\mathfrak{A}=(0,0,C)$ be a new $3-$core partition, and let g be a non-negative integer. Then we make the following definitions. I. The* **Type I** *generation g descendant of $\mathfrak{A}$ is the $3-$core whose abacus is of the form*

$$(0,g,C+g).$$

Denote this $3-$core by $I(g,C)$.

II. The **Type II** *generation g descendant of $\mathfrak{A}$ is the $3-$core whose abacus is of the form*

$$(0,C+g+1,g).$$

Denote this $3-$core by $II(g,C)$.

The following was given in [10].

Definition 5. *Let $\mathfrak{A} = (0, 0, C, D)$ be a new 4−core partition and let g be a non-negative integer. Then we make the following definitions.*
I. The **Type I** *generation g descendant of $\mathfrak{A}$ is the 4−core whose abacus is of the form*

$$(0, g, C + g, D + g).$$

Denote this 4−core by $I(g, C, D)$.
II. The **Type II** *generation g descendant of $\mathfrak{A}$ is the 4−core whose abacus is of the form*

$$(0, D + g + 1, g, C + g).$$

Denote this 4−core by $II(g, C, D)$.
III. The **Type III** *generation g descendant of $\mathfrak{A}$ is the 4−core whose abacus is of the form*

$$(0, C + g + 1, D + g + 1, g).$$

Denote this 4−core by $III(g, C, D)$.

Using these definitions, we now give structure theorems which give the parts of every 3−core and every 4−core. These results should be viewed as a generalization of the trivial observation that 2−cores are partitions whose parts are of the form

$$m,\ m - 1,\ m - 2, \ldots,\ 1.$$

For $t \geq 5$, there are similar structure theorems; however, they become very difficult to write down.

Theorem 2 (3− Core Structure Theorem). *Let $\mathfrak{A} = (0, 0, C)$ be a new 3−core partition and let $g \geq 0$ be a non-negative integer. Then we have:*
I. *The parts of the 3−core $I(g, C)$ are:*

$$\begin{aligned} &g + 2C, g + 2C - 2, \ldots, g + 2, \quad (C \text{ integers}) \\ &g, g, g - 1, g - 1, \ldots, 1, 1 \quad (g \text{ pairs}). \end{aligned}$$

II. *The parts of the 3−core $II(g, C)$ are:*

$$\begin{aligned} &g + 2C + 1, g + 2C - 1, \ldots, g + 1, \quad (C + 1 \text{ integers}) \\ &g, g, g - 1, g - 1, \ldots, 1, 1 \quad (g \text{ pairs}). \end{aligned}$$

Proof. The key observations needed for the proof are:

Observation 1. The smallest part of a partition is given by $\lambda_s = B_s$.

Observation 2. The difference between two consecutive structure numbers is

$$B_{i-1} - B_i = (\lambda_{i-1} - (i - 1) + s) - (\lambda_i - i + s) = \lambda_{i-1} - \lambda_i + 1.$$

Consequently, it is easy to see that

$$\lambda_{i-1} = (B_{i-1} - B_i) + \lambda_i - 1.$$

Now the parts of a $3-$core may be inductively obtained by starting from the smallest part $B_s = \lambda_s$ and then using the consecutive differences between two structure numbers to build the remaining parts.

We obtain the result by slicing each abacus into two pieces. The first slice consists of those rows of beads at the top of the abacus which contain beads exactly in columns 1 and 2. We shall refer to this slice as the $g-$**block.** The remaining slice consists of a single column of beads below the $g-$block. We shall refer to this slice as the $C-$**block.** Determine the parts of a $3-$core by examining the parts corresponding to these two blocks.

The $3-$core $I(g,C)$ (resp. $II(g,C)$) represents the abacus $(0,g,g+C)$ (resp. $(0,g+C+1,g)$). In either case, the top g rows of the abaci consisting of beads in the following positions

$$\begin{array}{cc}
(1,1), & (1,2) \\
(2,1), & (2,2) \\
(3,1), & (3,2) \\
. & . \\
. & . \\
. & . \\
(g-1,1), & (g-1,2) \\
(g,1), & (g,2)
\end{array}$$

form the $g-$block. Their structure numbers given by $B_i = 3(r_i - 1) + r'_i$ are:

$$\begin{array}{cc}
1 & 2 \\
4 & 5 \\
7 & 8 \\
. & . \\
. & . \\
. & . \\
3g-5 & 3g-4 \\
3g-2 & 3g-1.
\end{array}$$

By Observation 1, the smallest part in the $g-$block is 1. By Observation 2, the next smallest part size is $(2-1)+1-1 = 1$. The next smallest after that is $(4-2)+1-1 = 2$. Since the difference between consecutive structure numbers continues to alternate between 1's and 2's, the parts represented by consecutive beads of the $g-$block will alternately remain equal and differ by 1 in size. Continuing in this fashion, it is evident that the $g-$block represents the parts

$$g, g, g-1, g-1, \ldots, 2, 2, 1, 1.$$

Note that this is also valid for $g = 0$ since no parts are represented by the $g-$block in this case.

Immediately below the $g-$block is the $C-$block which consists of C beads in column 2 in the $I(g,C)$ case and $C+1$ beads in column 1 in the $II(g,C)$ case. The multiset elements coming from a single column of beads consist of a sequence of parts differing by 2 since beads in positions (r,r') and $(r+1,r')$ have the property that their structure numbers differ by 3. Hence, their part sizes will differ by 2.

In $I(g,C)$, the last bead in the first g rows is $(g,2)$ and the first bead of the following row is $(g+1,2)$. Since their structure numbers differ by 3, their part sizes differ by 2. It is now easy to verify that the initial sequence of beads in column 2 corresponds to the C parts

$$g+2C, g+2C-2, \ldots, g+4, g+2.$$

In a similar manner, since the first bead in row $g+1$ of $II(g, C)$ is $(g+1, 1)$, its structure number differs from that of $(g, 2)$ by 2. Thus the parts corresponding to the initial sequence of $C+1$ beads in column 1 are

$$g+2C+1, g+2C-1, \ldots, g+3, g+1.$$

□

The following theorem appears in [Th.5,10] and is proved in a similar manner.

Theorem 3 ($4-$**Core Structure Theorem).** *Let $\mathfrak{A} = (0, 0, C, D)$ be a new $4-$core partition and let $g \geq 0$ be a non-negative integer.*

I. *Define integers d and e by*

$$d := \min(C, D) \quad \text{and} \quad e := |C - D|.$$

a) If $C > D$, then the parts of the $4-$core $I(g, C, D)$ are:

$$g+2d+3e-1, g+2d+3e-4, \ldots, g+2d+2, \quad (e \text{ integers})$$
$$g+2d, g+2d, g+2d-2, g+2d-2, \ldots, g+2, g+2, \quad (d \text{ pairs})$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

b) If $C \leq D$, then the parts of the $4-$core $I(g, C, D)$ are:

$$g+2d+3e, g+2d+3e-3, \ldots, g+2d+3, \quad (e \text{ integers})$$
$$g+2d, g+2d, g+2d-2, g+2d-2, \ldots, g+2, g+2. \quad (d \text{ pairs})$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

II. *Define integers d and e by*

$$d := \min(2C+1, 2D+1) \quad \text{and} \quad e := |C - D|.$$

a) If $C > D$, then the parts of the $4-$core $II(g, C, D)$ are:

$$g+d+3e-2, g+d+3e-5, \ldots, g+d+1, \quad (e \text{ integers})$$
$$g+d, g+d-1, g+d-2, \ldots, g+1, \quad (d \text{ consecutive integers})$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

b) If $C \leq D$, then the parts of the $4-$core $II(g, C, D)$ are:

$$g+d+3e, g+d+3e-3, \ldots, g+d+3, \quad (e \text{ integers})$$
$$g+d, g+d-1, g+d-2, \ldots, g+1, \quad (d \text{ consecutive integers })$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

III. *Define integers d and e by*

$$d := \min(C+1, D+1) \quad \text{and} \quad e := |C - D|.$$

a) If $C > D$, then the parts of the $4-$core $III(g, C, D)$ are:

$$g+2d+3e-2, g+2d+3e-5, \ldots, g+2d+1, \quad (e \text{ integers})$$
$$g+2d-1, g+2d-1, g+2d-3, g+2d-3, \ldots, g+1, g+1, \quad (d \text{ pairs})$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

b) If $C \leq D$, then the parts of the $4-$core $III(g, C, D)$ are:

$$g+2d+3e-1, g+2d+3e-4, \ldots, g+2d+2, \quad (e \text{ integers})$$
$$g+2d-1, g+2d-1, g+2d-3, g+2d-3, \ldots, g+1, g+1, \quad (d \text{ pairs})$$
$$g, g, g, g-1, g-1, g-1, \ldots, 1, 1, 1 \quad (g \text{ triples}).$$

Before proceeding, we prove the following *conjugation* identities which are useful later in the article.

Definition 6. *If Λ is a partition whose parts are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, then its conjugate partition, denoted $\overline{\Lambda}$, is the unique partition whose Ferrers-Young diagram has λ_i nodes in column i.*

Let

$$\mathfrak{A}_1 \sim \mathfrak{A}_2$$

denote that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are abaci for conjugate partitions. Now we identify conjugate pairs of $3-$cores.

Proposition 1. *The following pairs of* 3*-core partitions are conjugates.*

$$\begin{aligned} &I(g,C) \sim I(C,g) \\ &II(g,C) \sim II(C,g). \end{aligned}$$

Proof. By the Structure Theorem for $3-$cores, a partition with abacus $I(g,C)$ has the following parts:

$$\begin{aligned} &g+2C, g+2C-2, \ldots, g+2, \quad (C \text{ integers}) \\ &g, g, g-1, g-1, \ldots, 1, 1 \quad (g \text{ pairs}). \end{aligned}$$

Thus, its column lengths are:

$$\begin{aligned} &C+2g, C+2(g-1), \ldots, C+2, \quad (g \text{ integers}) \\ &C, C, C-1, C-1, \ldots, 1, 1 \quad (C \text{ pairs}). \end{aligned}$$

These are the row sizes of $I(C,g)$. The second conjugation identity follows in exactly the same manner. $\square$

The following conjugation identities for $4-$cores appears in [Prop. 4,9].

Proposition 2. *Depending on type and on whether or not C is larger than D, the following pairs of partitions are conjugate.*

(i) If $D \geq C$, then:

$$\begin{aligned} &I(g,C,D) \sim I(D-C, C, C+g) \\ &II(g,C,D) \sim II(D-C, C, C+g). \end{aligned}$$

(ii) If $D < C$, then:

$$\begin{aligned} &II(g,C,D) \sim II(C-D-1, D+g+1, D) \\ &III(g,C,D) \sim III(C-D-1, D+g+1, D). \end{aligned}$$

(iii) If $D < C$, then

$$I(g,C,D) \sim III(C-D-1, D, D+g).$$

(iv) If $D \geq C$, then

$$III(g,C,D) \sim I(D-C, g+C+1, C).$$

3. Rook Equivalences

In this section, we examine the rook equivalence classes of $t-$core partitions. We shall speak of partitions and their associated Ferrers-Young diagrams (boards) interchangeably. But before we do this, we fix some notation.

If Λ is a partition of n with parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, augment Λ by setting $\lambda_i = 0$ for all $s < i \leq n$. This uniformization of the number of parts will make it easier to compare two different partitions of n. Define the *multiset* associated to Λ as the multiset of integers defined by

$$s_i = \lambda_i + i \tag{3}$$

for positive integers i. These multisets make it easy to determine whether or not two Ferrers boards are rook equivalent. As is common with multisets, let a^m denote m copies of the integer a.

Note that for purposes of notational convenience, we are using slightly non-standard notation. The standard method, as in Stanley [10], is to order partitions parts $\tilde{\lambda}_i$ in non-decreasing order (thus $\lambda_i = \tilde{\lambda}_{n-i+1}$) and to define multiset elements by $\tilde{s}_i = \tilde{\lambda}_i - i + 1$. We have done nothing more than to put a uniform translation on the multisets defined by the standard $\tilde{s}_i$ elements

$$\begin{aligned} s_i &= \lambda_i + i \\ &= \tilde{\lambda}_{n-i+1} + i \\ &= \tilde{\lambda}_{n-i+1} - (n - i + 1) + 1 + n \\ &= \tilde{s}_{n-i+1} + n. \end{aligned}$$

The following was proved by Goldman, Joichi, and White [6].

Proposition [Cor. 3, 6]. *Two Ferrers boards are rook equivalent if and only if their multisets are the same.*

Remark 2. A finite sequence of non-negative integers λ_i defines a Ferrers board, under the assumption that there are λ_i squares in row i, if and only if

$$\lambda_i \geq \lambda_{i+1}$$

for all i. Since $s_{i+1} - s_i = (\lambda_{i+1} + i + 1) - (\lambda_i + i) = \lambda_{i+1} - \lambda_i + 1$, it is easy to see that a sequence of integers s_i forms a multiset of a Ferrers board if and only if

$$s_{i+1} \leq s_i + 1 \text{ and } s_i \geq i.$$

We now prove the following theorem concerning the rook equivalence of $4-$cores. Then we will prove the analogous theorem for $3-$cores.

Theorem 4. *Two distinct* $4-$*core partitions are rook equivalent if and only if they are conjugates.*

Proof. To deduce that only conjugate $4-$cores have the same multiset, we shall employ Theorem 3 to construct and compare the multisets associated to $4-$cores. Let d and e be as defined in Theorem 3 and define the following *delta functions* by:

$$\delta_e(i) = \begin{cases} 1 & \text{if } i \leq e \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_g(i) = \begin{cases} 1 & \text{if } i \leq g \\ 0 & \text{otherwise.} \end{cases}$$

It will also be convenient to refer to the parts of a partition as grouped by Theorem 3 as the $e-$**block**, $d-$**block**, and $g-$**block**. Note that the structures of the e and g blocks do not vary in the sense that they consist of e distinct integers and g triples regardless of type.

Preliminary Observations. The following observations follow from Theorem 3 and (3). It is important to note that for every $4-$core; there is a unique positive integer x defined by either (ii) or (iii) of the following. It is important to note that x is always the smallest multiset element.

(i) The multiset elements coming from the $e-$block consist of a decreasing sequence of e integers with gaps of 2 between consecutive multiset elements.

There are two types of $d-$blocks.
(ii) One type of $d-$block consists of d pairs of parts with gaps of two between consecutive pairs. The multiset elements coming from such a block are

$$\{x^d, (x+1)^d\},$$

for some positive integer x.

However, by Theorem 3, when we have a type I partition with $d = 0$, this block is empty. To define x in this case, apply the following rules:

(1) If $g > 0$, then define x to be the smallest multiset element coming from the $g-$block.
(2) If $g = 0, e > 0$ and $C > D$, then let $x + 1$ be the smallest multiset element coming from the $e-$block.
(3) If $g = 0, e > 0$ and $C \leq D$, then let $x + 2$ be the smallest multiset element coming from the $e-$block.

It will be important to correctly 'glue' the $d-$block multiset to the multisets coming from the e and $g-$blocks. Note that when $d > 0$, one x is contributed by the $d-$block part adjacent to the e block, and one $(x+1)$ is contributed by the $d-$block part adjacent to the g block.

(iii) The second type of $d-$block consists of d consecutive parts. These parts contribute x^d to the multiset for some positive integer x.
(iv) The $g-$block consists of the g triples

$$g, g, g, g-1, g-1, g-1, \dots, 1, 1, 1.$$

Now append infintely many parts of size 0 to the $g-$block

$$g, g, g, g-1, g-1, g-1, \dots, 1, 1, 1, 0, 0, 0, 0, 0, 0, \dots \quad .$$

This is legal since parts of size zero do not effect rook equivalences. Hence, these parts contribute multisets of the form

$$\{y, y+1, (y+2)^2, (y+3), (y+4)^2, \cdots, (y+2g-2)^2, y+2g-1, (y+2g)^2\}$$

$$\cup \{y+2g+1, y+2g+2, y+2g+3, \dots\}$$

for some positive integer y.

Since every $4-$core partition can be sliced into these blocks, determining the associated multisets is simply a matter of *gluing* the resulting multisets from $(i), (ii), (iii)$, and (iv), together in the correct manner. We now consider each of the six types of partitions given by Theorem 3.

Case *I***:** If $d>0$ then by (ii) the multiset elements coming from the $d-$block are x^d and $(x+1)^d$. The smallest $d-$block part, $g+2$, produces the multiset element $x+1$ and is adjacent to the largest $g-$block part, g. Thus the multiset element coming from g is x. This is the smallest element of the multiset coming from the $g-$block.

Subcase (a): Since the largest part of the $d-$block and the smallest part of the $e-$block differ by 2, the smallest multiset element coming from the $e-$block is $x+1$. Note that $e>0$ for all such partitions.

Subcase (b): Since the largest part of the $d-$block and the smallest part of the $e-$block differ by 3, the smallest multiset element coming from the $e-$block will be $x+2$.

If $d=0$, then the smallest multiset element x comes from the $g-$block. Furthermore, the smallest multiset elements coming from the $e-$block, if it exists, are still $x+1$ and $x+2$ in Subcase (a) and Subcase (b) respectively.

Case *II***:** The d-block multiset elements are x^d. The smallest $d-$block part and the largest $g-$block part differ by 1, hence the smallest contribution of the $g-$block to the multiset is x.

Subcase (a): Since the largest $d-$block part and the smallest $e-$block part differ by 1, the smallest $e-$block multiset element is x. Note that $e>0$ for all such partitions.

Subcase (b): Since the largest $d-$block part and the smallest $e-$block part differ by 3, the smallest multiset element coming from the $e-$block is $x+2$.

Case *III:* The multiset elements coming from the d-block are x^d and $(x+1)^d$. Since the smallest $d-$block part and the largest $g-$block part differ by 1, the smallest multiset element coming from the $g-$block is $(x+1)$.

Subcase (a): Since the largest $d-$block part and the smallest $e-$block part differ by 2, the smallest multiset element coming from the $e-$block is $x+1$. Note that $e>0$ for all partitions included in this subcase.

Subcase (b): Since the largest $d-$block part and the smallest $e-$block part differ by 3, the smallest multiset element coming from the $e-$block is $x+2$.

The following tables describe the multisets for each subtype of $4-$core. The multisets are listed relative to the smallest multiset member x. Specifically, by the *multiplicity of value* i we mean the multiplicity of $x+i-1$. The δ_e*'s* and δ_g*'s* are used to account for the contribution given by the $e-$ and $g-$blocks.

(4)

Type	Value 1	Value 2	Value 3
$I(a)$	$d+1$	$d+\delta_e(1)+1$	$\delta_g(1)+1$
$I(b)$	$d+1$	$d+1$	$\delta_e(1)+\delta_g(1)+1$
$II(a)$	$d+\delta_e(1)+1$	1	$\delta_e(2)+\delta_g(1)+1$
$II(b)$	$d+1$	1	$\delta_e(1)+\delta_g(1)+1$
$III(a)$	d	$d+\delta_e(1)+1$	1
$III(b)$	d	$d+1$	$\delta_e(1)+1.$

More generally, the values for $m\geq 2$ are given by:

(5)

Type	Value $2m$	Value $2m$+1
$I(a)$	$\delta_e(m)+1$	$\delta_g(m)+1$
$I(b)$	1	$\delta_e(m)+\delta_g(m)+1$
$II(a)$	1	$\delta_e(m+1)+\delta_g(m)+1$
$II(b)$	1	$\delta_e(m)+\delta_g(m)+1$
$III(a)$	$\delta_e(m)+\delta_g(m-1)+1$	1
$III(b)$	$\delta_g(m-1)+1$	$\delta_e(m)+1.$

We now show how the table entries were computed for the type $II(a)$ partitions. The multiset contributions given by the e, d, g-blocks, and the infinite number of parts of size 0 which were appended to the $g-$block respectively are:

$$
\begin{aligned}
&x, x+2, x+4, \ldots, x+2(e-1),\\
&x^d,\\
&x, x+1, x+2, x+2, x+3, x+4, \ldots, x+2g-2, x+2g-1, x+2g,\\
&x+2g, x+2g+1, x+2g+2, x+2g+3, \ldots \quad .
\end{aligned}
$$

Adding up the multiplicites of this set gives row $II(a)$ of (4) and (5). The other rows follow similarly.

Now we show that two different rook-equivalent $4-$cores are conjugates.

Case *I(a):* First, we examine which $4-$cores are rook equivalent to a $I(a)$ partition. Recall that in Subcase $I(a)$ it is known that $e > 0$. Therefore, by (4), it is easy to see that a $I(a)$ is not rook equivalent to any $4-$core of type $I(b)$ since the multiplicities of values 1 and 2 are equal for a $I(b)$ but are unequal for a $I(a)$. Moreover, since the multiplicites of $I(a)$ are uniquely determined by the e, d and g, it easily follows that a $I(a)$ is not rook equivalent to another $I(a)$ partition.

By (4), the multiplicities of the two smallest multiset elements are $d+1$ and $d+2$ respectively. The only other partition with this property is a $III(b) = III(g', C', D')$ where $d' = d+1$. This is the case since $e > 0$ for $III(a)$ partitions. Therefore,

$$\min(C, D) + 1 = \min(C' + 1, D' + 1).$$

But since $C > D$ and $C' \leq D'$, this means that

$$C' = D.$$

Moreover, by (4) and (5), it follows that $e' = D' - C' = g$ and $g' = e - 1 = C - D - 1$. Therefore it is easy to see that $D' = g + D$ and $g' = C - D - 1$. Hence the $III(b)$ partition is $III(C-D-1, D, D+g)$. By Proposition 2, these are conjugate partitions.

Case *I(b):* Since the first two multiplicities of a $I(b)$ are equal and $e > 0$ in a $I(a)$, a I(b) can only be rook equivalent to another $I(b)$. If this is the case, then by the symmetry of the formulae in (4) and (5) the multisets of two different $I(b)$ partitions are the same if and only if the two partitions have switched e and g values. But by Proposition 2 and Theorem 3 such partitions must be conjugate.

Case *II(a)***:** Since $e > 0$, the first two multiplicites of the multiset of a $II(a)$ are $d+2$ and 1. Since d is odd, the multiplicity of value 1 is at least 3. Hence, any rook equivalent $4-$core must be another $II(a)$ $4-$core. The only other $II(a)$ $4-$core with the same multiset is found by setting $d' = d, e' = g+1$ and $g' = e-1$. It is easy to verify that these $4-$cores are conjugates.

Case *II(b)***:** The multiplicities of the two smallest values in the multiset are $d+1$ and 1 where d is odd. So it is easy to see that the only $4-$cores that are possibly rook equivalent to a $II(b)$ is a $II(a)$ or a $II(b)$. However, it is impossible for a $II(b)$ and a $II(a)$ to be rook equivalent because the multiplicities of value 1 are of opposite parities. Therefore, a type $II(b)$ can only be rook equivalent to another $II(b)$ where at most the e and g are switched. However, it is easy to verify that this switch is equivalent to conjugation.

Case *III(a)***:** Since $e > 0$, the multiplicities of the two smallest multiset elements are d and $d+2$. There are no other types with this property. Hence, any rook equivalent $4-$core must be another $III(a)$ $4-$core. The only other $III(a)$ $4-$core with the same multiset is found by setting $d' = d, e' = g+1$ and $g' = e-1$. It is easy to verify that this represents a conjugation.

Case *III(b):* The multiplicities of the two smallest elements in the multiset are d and $d+1$. The only different $4-$core with this property is a $I(a)$ where $d' = d-1, g' = e$, and $e' = g+1$. However, this partition is conjugate to the given $III(b)$ partition.

This completes the proof of this Theorem.

□

Theorem 5. *Two distinct* 3*-core partitions are rook equivalent if and only if they are conjugates.*

Proof. Using the methods above, we obtain the following table for multiplicities of multisets.

Type	Value 1	Value 2	...	Value m
$I(g,C)$	1	$1+\delta_C(1)+\delta_g(1)$	...	$1+\delta_C(m-1)+\delta_g(m-1)$
$II(g,C)$	2	$1+\delta_C(1)+\delta_g(1)$	...	$1+\delta_C(m-1)+\delta_g(m-1)$.

Clearly, the only non-trivial rook-equivalencies that occur are those that are obtained by switching C and g. However, by Proposition 1, it is easy to see that this is simply conjugation.

□

As a corollary to Remark 1, Theorem 4 and Theorem 5, we obtain the Main Theorem.

Main Theorem. *If $t = 2, 3$ or 4, then two distinct $t-$core partitions are rook equivalent if and only if they are conjugates.*

We now show that the situation is very different if $t \geq 5$. In particular, there exists rook equivalent $t-$cores which are not conjugates for certain $n < t$.

Theorem 6. *If $t \geq 5$, then there are pairs of distinct rook equivalent $t-$core partitions which are not conjugates.*

Proof. If n is a positive integer for which every pair of rook equivalent $t-$cores is conjugate, then it is clear that

$$c_t(n) = 2a_t(n) - sc_t(n),$$

where $a_t(n)$ is the number of rook equivalence classes of Ferrers boards of size n containing a $t-$core, and $sc_t(n)$ equals the number of self conjugate $t-$cores of n.

However, it is clear that if $t > n$, then

$$c_t(n) = p(n),$$

since a partition with fewer than t nodes cannot have a $t-$hook. Hence, the theorem of Foata and Schützenberger implies that

$$a_t(n) \leq q(n).$$

In particular, we find that if $t > n$, then

$$p(n) = c_t(n) = 2a_t(n) - sc_t(n) \leq 2q(n).$$

Since $p(4) = 5$ and $q(4) = 2$, we get an immediate contradiction if $t \geq 5$. □

However, much more is probably true. Computational evidence suggests the following which pertains to those $n > t$.

Conjecture. *If $t \geq 5$, then there exists a constant $N(t)$ with the property that if $n \geq N(t)$, then there exist two distinct rook equivalent $t-$cores of size n which are not conjugates.*

4. A finer investigation

In this section, we investigate the rook equivalence classes which contain $t-$cores when $t = 2, 3,$ and 4. In [10] the following theorem was proved.

Theorem [Th.1,10]. *If n is a non-negative integer for which $8n + 5$ is square-free, then*

$$c_4(n) = \frac{1}{2}h(-32n - 20).$$

By Theorem 4 we obtain the following immediate corollary:

Corollary 1. *Let $a_4(n)$ denote the number of rook equivalence classes of Ferrers boards of size n containing a $4-$core, and let $sc_4(n)$ denote the number of self-conjugate $4-$cores of n. If n is a non-negative integer for which $8n + 5$ is square-free, then*

$$h(-32n - 20) = 4a_4(n) - 2sc_4(n). \tag{6}$$

Proof. By the above theorem, it is known that

$$h(-32n - 20) = 2c_4(n).$$

However, by Theorem 4, it is known that

$$c_4(n) = 2a_4(n) - sc_4(n).$$

□

Remark 3. By [Th. 3,10], we have an explicit formula for $sc_4(n)$. If n is a non-negative integer whose factorization into distinct primes $p_i \equiv 1 \pmod 4$ and $q_j \equiv 3 \pmod 4$ is

$$8n + 5 = \prod {p_i}^{\alpha_i} q_j^{\beta_j},$$

then

$$sc_4(n) = \begin{cases} 0 & \text{if any } \beta_j \equiv 1 \pmod 2, \\ \frac{1}{2}\prod(\alpha_i + 1) & \text{otherwise.} \end{cases}$$

Therefore, the real mystery is how to compute $a_4(n)$. Typically, the difficulty in computing class numbers boils down to special properties of $L-$functions or to problems dealing with the explicit construction of elements in the class group. For discriminants of the form $-32n - 20$, explicitly constructing $4-$cores of n is equivalent to constructing elements in the class group, as is shown in [10]. Therefore, it may appear as if there is no advantage to this combinatorial interpretation of the class group.

The principal advantage we have when working with these combinatorial structures is that with the additional notion of rook equivalence, we obtain a new criterion for establishing the existence of elements in the class group. Since rook equivalence classes containing a $4-$core typically contain many partitions, we no longer need to construct $4-$cores to obtain large class numbers; we simply need to detect the existence of partitions that are rook equivalent to $4-$cores. Goldman, Joichi, and White proved [6] the following theorem which determines the number of partitions that are rook equivalent to any given board.

Theorem [Th.6,6]. *Given a Ferrers board B, append an infinite number of parts of size zero to B, and define the multiset $s_1, s_2, \ldots$ as in section 3. Define non-negative integers a_i by*

$$a_i = \#\{j \mid s_j = i\}.$$

Let $b = \min\{i \mid a_i > 0\}$ and $c = \max\{i \mid a_i > 1\}$. The number of Ferrers boards (with rows of non-zero size) rook equivalent to B is

$$\prod_{i=b}^{c-1} \binom{a_i + a_{i+1} - 1}{a_i}. \tag{7}$$

The reader should be aware that our formulation of this theorem is slightly different from the original formulation. This follows from the fact that our multiset elements are defined in a slightly different manner as was explained earlier. It is an easy exercise to verify that the above formulation agrees with other treatments. As an immediate Corollary, we obtain the following:

Corollary 2. *If Λ is a partition, then let $N(\Lambda)$ denote the number of partitions rook equivalent to Λ. If Λ is a $t-$core with $t = 2, 3$, or 4, then $N(\Lambda)$ is given by the following formulae:*

1. *If Λ is a $2-$core, then $N(\Lambda) = 1$.*
2. *Let $u = \min(C, g)$, and $v = \max(C, g)$. If $\Lambda = I(g, C)$ is a $3-$core, then*
$$N(\Lambda) = (1 + \delta_C(1) + \delta_g(1))\, 10^{\max(0,u-1)} 4^{\min(1,v-u,u)} 3^{\max(0,v-u-1)}.$$
3. *Let $u = \min(C, g)$, and $v = \max(C, g)$. If $\Lambda = II(g, C)$ is a $3-$core, then*
$$N(\Lambda) = \binom{2 + \delta_C(1) + \delta_g(1)}{2} 10^{\max(0,u-1)} 4^{\min(1,v-u,u)} 3^{\max(0,v-u-1)}.$$
4. *If $\Lambda = I(g, C, D)$ is a $4-$core with $C > D$ and $C - D \geq g + 1$, then*
$$N(\Lambda) = \binom{2D+2}{D+1} \binom{D + 2 + \delta_g(1)}{D+2} 3^{\max(0,2g-1)} 2^{C-D-g-1}.$$
5. *If $\Lambda = I(g, C, D)$ is a $4-$core with $C > D$ and $C - D \leq g$, then*
$$N(\Lambda) = \binom{2D+2}{D+1} \binom{D+3}{D+2} 3^{2C-2D-2} 2^{g-C+D}.$$
6. *Let $u = \min(D - C, g)$ and $v = \max(D - C, g)$. If $\Lambda = I(g, C, D)$ is a $4-$core with $D \geq C$, then*
$$N(\Lambda) = \begin{cases} \binom{2C+1}{C+1} & if\ u = v = 0 \\ \binom{2C+1}{C+1}\binom{C+2}{C+1} 2^{v-1} & if\ u = 0 \text{ and } v > 0 \\ \binom{2C+1}{C+1}\binom{C+3}{C+1} 3^{u-1} 2^{v-u} & if\ u > 0. \end{cases}$$
7. *Let $u = \min(C - D, g+1)$ and $v = \max(C - D, g+1)$. If $\Lambda = II(g, C, D)$ is a $4-$core with $C > D$, then*
$$N(\Lambda) = 3^{u-1} 2^{v-u}.$$
8. *Let $u = \min(D - C, g)$ and $v = \max(D - C, g)$. If $\Lambda = II(g, C, D)$ is a $4-$core with $D \geq C$, then*
$$N(\Lambda) = 3^u 2^{v-u}.$$

(9) *Let* $u = \min(C-D, g+1)$ *and* $v = \max(C-D, g+1)$. *If* $\Lambda = III(g,C,D)$ *is a* $4-$*core with* $C > D$, *then*

$$N(\Lambda) = \binom{2D+3}{D+1} 3^{u-1} 2^{v-u}.$$

(10) *If* $\Lambda = III(g,C,D)$ *is* $4-$*core with* $D \geq C$ *and* $D - C \geq g+1$, *then*

$$N(\Lambda) = \binom{2C+2}{C+1}(C+3)3^{2g}2^{D-C-g-1}.$$

(11) *If* $\Lambda = III(g,C,D)$ *is a* $4-$*core with* $D \geq C$ *and* $D - C \leq g$, *then*

$$N(\Lambda) = \binom{2C+2}{C+1}\binom{C+2+\delta_{D-C}(1)}{C+2} 3^{\max(0,2D-2C-1)} 2^{g+1-D+C}.$$

Proof. This corollary follows easily from Remark 1, Tables (4), and (5), formula (7). We demonstrate the proof in cases (4) and (5). Let $u = \min(2(C-D), 2g+1)$ and $v = \max(2(C-D), 2g+1)$. By tables (4) and (5), the multiplicities of the multiset for $I(g,C,D)$ with $C > D$ consist of D, $D+1$, followed by a string of 2's from Value 3 to Value $u+1$, and ending with alternating 1's and 2's from Value $u+2$ through Value v. Thus, the first two factors of (7) in [Th. 6,6] are

$$\binom{2D+2}{D+1} \text{ and } \binom{D+2+\delta_g(1)}{D+2}.$$

The product of the remaining non-trivial factors is

$$\binom{2+2-1}{2}^{\max(0,u-2)} \binom{1+2-1}{1}^{\frac{v-u-1}{2}}.$$

Breaking u and v into cases yields cases (4) and (5). □

Consequently, it is easy to see that on average the number of partitions rook equivalent to any given $4-$core is fairly large. Therefore, it is desirable to obtain an algorithm or general method which detects any such partition. In particular, if $p = 8n+5$ is prime, then finding a single partition rook equivalent to a non-self-conjugate $4-$core implies that $h(-32n-20) \geq 6$. We should note that if $p \equiv 5 \pmod{24}$, then such a method exists, and so it is known that $h(-4p) \geq 6$ for such p. This is discussed in [10].

For completeness, we will list the distinct parts of partitions which are rook equivalent to any $t-$core for $t = 2, 3$, and 4. This follows as a corollary to Theorems 2 and 3.

Corollary 3. *If* Λ *is a* $t-$*core with* $t = 2, 3,$ *or* 4, *then the unique partition into distinct parts rook equivalent to* Λ, *which we denote by* $\tilde{\Lambda}$, *is given by the following rules:*

(1) *If* Λ *is a* $2-$*core, then* $\tilde{\Lambda} = \Lambda$.
(2) *Let* $u = \min(C, g)$ *and* $v = \max(C, g)$. *If* $\Lambda = I(g,C)$, *then* $\tilde{\Lambda}$ *is*

$$u+2v, u+2v-2, u+2v-4, \ldots, 3u+2, \quad (v-u \text{ integers})$$
$$3u, 3u-1, 3u-3, 3u-4, \ldots, 6, 5, 3, 2 \quad (2u \text{ integers}).$$

(3) *Let* $u = \min(C, g)$ *and* $v = \max(C, g)$. *If* $\Lambda = II(g, C)$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&u + 2v + 1, u + 2v - 1, u + 2v - 3, \ldots, 3u + 3, \quad (v - u)\\ &3u + 1, 3u,\ 3u - 2, 3u - 3, \ldots, 4, 3,\ 1 \quad (2u + 1).\end{aligned}$$

(4) *If* $\Lambda = I(g, C, D)$, *with* $C > D$ *and* $C - D \geq g$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3C - D + g - 1, 3C - D + g - 4, \ldots, 2D + 4g + 2, \quad (C - D - g)\\ &2D + 4g, 2D + 4g - 2, 2D + 4g - 4, \ldots, 2D + 2, \quad (2g)\\ &2D + 1, 2D, 2D - 1, \ldots, D + 2, \quad (D)\\ &D, D - 1, D - 2, \ldots, 1 \quad (D).\end{aligned}$$

(5) *If* $\Lambda = I(g, C, D)$, *with* $C > D$ *and* $C - D < g$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g + D + C, 3g + D + C - 3, \ldots, 4C - 2D + 3, \quad (g - C + D)\\ &4C - 2D, 4C - 2D - 2, 4C - 2D - 4, \ldots, 2D + 2, \quad (2C - 2D)\\ &2D + 1, 2D, 2D - 1, \ldots, D + 2, \quad (D)\\ &D, D - 1, D - 2, \ldots, 1 \quad (D).\end{aligned}$$

(6) *If* $\Lambda = I(g, C, D)$, *with* $D \geq C$ *and* $D - C \geq g$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3D + g - C, 3D + g - C - 3, \ldots, 2C + 4g + 3, \quad (D - C - g)\\ &4g + 2C, 4g + 2C - 1,\ 4g + 2C - 4, 4g + 2C - 5, \ldots,\ 2C + 4, 2C + 3, \quad (2g)\\ &2C + 1, 2C, 2C - 1, \ldots, C + 2, \quad (C)\\ &C, C - 1, C - 2, \ldots, 1 \quad (C).\end{aligned}$$

(7) *If* $\Lambda = I(g, C, D)$, *with* $D \geq C$ *and* $D - C < g$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g + D + C, 3g + D + C - 3, \ldots, 4D - 2C + 3, \quad (g - D + C)\\ &4D - 2C, 4D - 2C - 1,\ 4D - 2C - 4, 4D - 2C - 5, \ldots,\ 2C + 4, 2C + 3, \quad (2D - 2C)\\ &2C + 1, 2C, 2C - 1, \ldots, C + 2, \quad (C)\\ &C, C - 1, C - 2, \ldots, 1 \quad (C).\end{aligned}$$

(8) *If* $\Lambda = II(g, C, D)$, *with* $C > D$ *and* $C - D \geq g + 1$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3C - D + g - 1, 3C - D + g - 4, \ldots, 2D + 4g + 5, \quad (C - D - g - 1)\\ &2D + 4g + 2, 2D + 4g + 1,\ 2D + 4g - 2, 2D + 4g - 3, \ldots,\ 2D + 2, 2D + 1, \quad (2g + 2)\\ &2D, 2D - 1, 2D - 2, \ldots, 1 \quad (2D).\end{aligned}$$

(9) *If* $\Lambda = II(g, C, D)$, *with* $C > D$ *and* $C - D \leq g$, *then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g + D + C + 1, 3g + D + C - 2, \ldots, 4C - 2D + 1, \quad (g - C + D + 1)\\ &4C - 2D - 2, 4C - 2D - 3,\ 4C - 2D - 6, 4C - 2D - 7, \ldots, 2D + 2, 2D + 1, \quad (2C - 2D)\\ &2D, 2D - 1, 2D - 2, \ldots, 1 \quad (2D).\end{aligned}$$

(10) *If* $\Lambda = II(g,C,D)$*, with* $D \geq C$ *and* $D - C \geq g$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3D+g-C+1, 3D+g-C-2, \ldots, 2C+4g+4, \quad (D-C-g)\\&4g+2C+1, 4g+2C,\ 4g+2C-3, 4g+2C-4, \ldots, 2C+5, 2C+4, \quad (2g)\\&2C+1, \ldots, 3, 2, 1 \quad (2C+1).\end{aligned}$$

(11) *If* $\Lambda = II(g,C,D)$*, with* $D \geq C$ *and* $D - C < g$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g+D+C+1, 3g+D+C-2, \ldots, 4D-2C+4, \quad (g-D+C)\\&4D-2C+1, 4D-2C,\ 4D-2C-3, 4D-2C-4, \ldots, 2C+5, 2C+4, \quad (2D-2C)\\&2C+1, \ldots, 3, 2, 1 \quad (2C+1).\end{aligned}$$

(12) *If* $\Lambda = III(g,C,D)$*, with* $C > D$ *and* $C - D \geq g+1$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3C-D+g, 3C-D+g-3, \ldots, 2D+4g+6, \quad (C-D-g-1)\\&2D+4g+3, 2D+4g+2,\ 2D+4g-1, 2D+4g-2, \ldots, 2D+3, 2D+2, \quad (2g+2)\\&2D+1, 2D, 2D-1, \ldots, D+2, \quad (D)\\&D, D-1, D-2, \ldots, 1 \quad (D).\end{aligned}$$

(13) *If* $\Lambda = III(g,C,D)$*, with* $C > D$ *and* $C - D \leq g$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g+D+C+2, 3g+D+C-1, \ldots, 4C-2D+2, \quad (g-C+D+1)\\&4C-2D-1, 4C-2D-2,\ 4C-2D-5, 4C-2D-6, \ldots,\ 2D+3, 2D+2, \quad (2C-2D)\\&2D+1, 2D, 2D-1, \ldots, D+2, \quad (D)\\&D, D-1, D-2, \ldots, 1 \quad (D).\end{aligned}$$

(14) *If* $\Lambda = III(g,C,D)$*, with* $D \geq C$ *and* $D - C \geq g+1$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3D+g-C+1, 3D+g-C-2, \ldots, 2C+4g+7, \quad (D-C-g-1)\\&4g+2C+4, 4g+2C+2, 4g+2C, \ldots, 2C+4, \quad (2g+1)\\&2C+2, 2C+1, 2C, \ldots, C+2, \quad (C+1)\\&C, C-1, C-2, \ldots, 1 \quad (C).\end{aligned}$$

(15) *If* $\Lambda = III(g,C,D)$*, with* $D \geq C$ *and* $D - C \leq g$*, then* $\tilde{\Lambda}$ *is*

$$\begin{aligned}&3g+D+C+2, 3g+D+C-1, \ldots, 4D-2C+5, \quad (g-D+C)\\&4D-2C+2, 4D-2C, 4D-2C-2, \ldots, 2C+4, \quad (2D-2C)\\&2C+2, 2C+1, 2C, \ldots, C+2, \quad (C+1)\\&C, C-1, C-2, \ldots, 1 \quad (C).\end{aligned}$$

Proof. Suppose $\tilde{\Lambda}$ is a partition composed of s distinct parts. List the parts sizes of $\tilde{\Lambda}$ in decreasing order. Augment this list by an infinite number of zeros. Create its multiset elements, s_i, by adding i to the ith part size for each $i \geq 1$. The s_i are non-increasing up through the first part, $s+1$, of size zero, after which the s_i increase by one each time. Thus, the first part of size zero produces the smallest multiset element $x = s + 1$ of $\tilde{\Lambda}$. Clearly, given a multiset, we can reverse this process to produce a partition into distinct parts that belongs to that multiset. This procedure is described in steps 2), 3), and 4) of the following algorithm. We now state the algorithm for constructing a partition into distinct parts that is rook equivalent to a $t-$core, for $2 \leq t \leq 4$:

1) Given one of the $t-$core partitions Λ described in the theorem, construct its multiset.

2) Define x to be the smallest multiset element.

3) Throw out one copy of each of the elements in the multiset (reduce the multiplicity of each element by one). Since we wish to produce a partition into distinct parts, this process removes the multiset elements that will come from the parts of size zero of such a partition. Let s be the finite number of elements in this new multiset.

4) List the elements of the new multiset in non-decreasing order. Subtract s from the first one, $s - 1$ from the second, etc. After doing these subtractions, we are left with the partition, $\tilde{\Lambda}$, composed of distinct parts belonging to the multiset of step 2).

Since Λ and $\tilde{\Lambda}$ have the same multiset, they are rook equivalent [Corollary 3, 6].

For example, say Λ is a $4-$core partition of type $II(a)$ as described in Theorem 3. Performing steps 1), 2) and 3) above, using Tables (4) and (5), our new multiset becomes

$$(g+C+D+2)^{2D+1}(g+C+D+2)^{\delta_e(1)}(g+C+D+4)^{\delta_e(2)+\delta_g(1)}(g+C+D+6)^{\delta_e(3)+\delta_g(2)}\cdots . \tag{8}$$

The smallest element of this multiset is $x = g + C + D + 2$. The number of non-zero parts in the partition into distinct parts that produces this multiset is given by $s = x - 1 = g + C + D + 1$. Thus, to perform step 4) on (8), we subtract $g + C + D + 1$ from the first element $g + C + D + 2$, then subtract $g + C + D$ from the second element (which will also be $g + C + D + 2$ if $D > 0$), etc. To describe the final result, we need to consider the cases $C - D \geq g + 1$ and $C - D \leq g$ separately. Finally, rearranging our partition elements into non-decreasing order results in the formulas listed in Corollary 4 corresponding to $4-$core partitions of type $II(a)$. The same procedure is used for the other cases.

□

Example 2. Consider the type $II(b)$ $4-$core partition Λ with $C = 1$, $D = 2$, and $g = 1$. By Theorem 3, the parts of Λ are 7 , 4, 3, 2, 1, 1, 1, while by Corollary 4 the parts of the rook equivalent partition into distinct parts are 7,6,3,2,1.

Acknowledgement. The authors would like to thank Charles Grassl of CRAY Research (Mendota Heights, Minnesota) for running a FORTRAN program which computed the number of rook equivalence classes of $t-$cores. His computations were done using a CRAY C-90.

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Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: `jhaglund@math.uiuc.edu`

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540
E-mail address: `ono@math.ias.edu`

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802
E-mail address: `ono@math.psu.edu`

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802
E-mail address: `sze@math.psu.edu`